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O. G. NIKOLAEV, M. V. SKITSKA, N. V. DRASHPUL

OPTIMAL CONTROL OF THE THERMOELASTIC STATE OF A BALL WITH A SPHERICAL INCLUSION USING POWER OF DISTRIBUTED INTERNAL HEAT SOURCES

In various fields of engineering and technology, problems arise in creating various types of technical systems with internal heat sources. Their design is based on modeling and optimization of the obtained models, which cannot always be achieved with guaranteed accuracy and completeness using known numerical methods. Therefore, further development of a new effective numerical-analytical method for solving similar problems for composite bodies with complex geometry and arbitrary thermomechanical characteristics of the constituent components, proposed by the authors in one of the previous articles, is an actual task. This is the focus of this paper. Using the example of an elastic body with a certain geometry (a ball with an eccentric spherical inclusion), the problem of optimally controlling its thermoelastic steady- state is solved using the power of distributed heat sources within the inclusion. The objective functional in the problem was chosen to be the functional that expresses the root-mean-square value of the stress on the inclusion surface. An additional limitation is imposed on the mean square power of heat sources. This formulation of the problem is considered for the first time. The solution to the problem is divided into two stages. The first stage solves the direct problem of modeling the thermoelastic state of a ball with an inclusion at a given temperature and external load on its surface, as well as the power of distributed heat sources in the inclusion. In this case, the generalized Fourier method is used, the modified apparatus of which for the specified body geometry was developed by the authors in the work. As a result of the implementation of the first stage, the original problem is replaced by an equivalent problem of optimal control of the state of an object, which is defined by two infinite linear algebraic systems. In this case, the optimization problem is posed with respect to a quadratic functional defined on the Cartesian product of Hilbert spaces of square-summable numerical sequences. The arguments of the functional are the solutions of the specified systems. The fundamental difficulty in solving the equivalent (inverse) problem is the impossibility of analytically expressing the solutions of systems through optimization parameters. The work further develops the method of parametric solution of infinite systems of linear algebraic equations proposed by the authors. As a result of its application, the quadratic functional is expressed in terms of optimization parameters. The problem of finding a conditional extremum of a functional is solved using the Lagrange method. Its application leads to an infinite system of linear algebraic equations with a quadratic constraint, which is solved by the spectral method. All intermediate stages of the implementation of the proposed method are strictly justified by five theorems proven in the work. The numerical implementation of the method was carried out within the framework of an extensive computer experiment, the results of which are presented in the paper. The work shows pictures of optimal temperature fields in a ball and the powers of distributed heat sources in the inclusion, as well as graphs of optimal temperatures and normal stresses on the surface of the inclusion depending on the geometric parameters and three different types of external load. All the main results of the work are new. The analytical justification and calculations performed prove the correctness and effectiveness of the proposed method.

Key words: optimal control, thermoelastic state, power of distributed heat sources, objective functional, modeling, generalized Fourier method, infinite system of linear algebraic equations, parametric solution, rigorous justification.

О. Г. НІКОЛАЄВ, М. В. СКІЦКА, Н. В. ДРАШПУЛЬ

ОПТИМАЛЬНЕ КЕРУВАННЯ ТЕРМОПРУЖНИМ СТАНОМ КУЛІ ЗІ СФЕРИЧНИМ ВКЛЮЧЕННЯМ ЗА ДОПОМОГОЮ ПОТУЖНОСТІ РОЗПОДІЛЕНИХ ВНУТРІШНІХ ТЕПЛОВИХ ДЖЕРЕЛ

У різних галузях техніки та технології виникають задачі створення різноманітних типів технічних систем із внутрішніми джерелами тепла. Їх конструювання ґрунтується на моделюванні та оптимізації отриманих моделей, що не завжди можна виконати з гарантованою точністю та повнотою відомими чисельними методами. Тому подальший розвиток нового ефективного чисельно-аналітичного методу розв'язання подібних задач для складових тіл зі складною геометрією та довільними термомеханічними характеристиками складових компонентів, запропонованого авторами в одній із попередніх статей, є актуальною задачею. У цьому і полягає суть даної роботи. На прикладі пружного тіла з певною геометрією (куля з ексцентричним сферичним включенням) розв'язується задача оптимального керування його стаціонарним термопружним станом за допомогою потужності розподілених джерел тепла у включенні. Цільовим функціоналом у задачі було обрано функціонал, який виражає середньоквадратичне значення напруження на поверхні включення. Додаткове обмеження накладається на середній квадрат потужності теплових джерел. Така постановка задачі розглядається вперше. Розв'язання задачі розбите на два етапи. На першому – розв'язується пряма задача моделювання термопружного стану кулі з включенням при заданих на межі кулі температурі та зовнішньому навантаженні, а у включенні – потужності розподілених теплових джерел. При цьому використовується узагальнений метод Фур'є, модифікований апарат якого для зазначеної геометрії тіла розроблено авторами в роботі. В результаті реалізації першого етапу вихідна задача замінена на еквівалентну задачу оптимального керування станом об'єкта, який задається двома нескінченними лінійними алгебраїчними системами. При цьому оптимізаційна задача ставиться відносно квадратичного функціонала, визначеного на декартовому добутку гільбертових просторів числових послідовностей, сумовних з квадратом. Аргументами функціонала є розв'язки вказаних систем. Принциповою складністю розв'язання еквівалентної (зворотної) задачі є неможливість аналітичного вираження розв'язків систем через параметри оптимізації. У роботі отримав подальший розвиток запропонований авторами метод параметричного розв'зання нескінченних систем лінійних алгебраїчних рівнянь. Внаслідок його застосування квадратичний функціонал виражається через параметри оптимізації. Задача на умовний екстремум функціонала вирішується методом Лагранжа. Його застосування призводить до нескінченної системи лінійних алгебраїчних рівнянь з квадратичним обмеженням, яка розв'язується спектральним методом. Усі проміжні етапи реалізації запропонованого методу строго обґрунтовані п'ятьма доведеними у роботі теоремами. Чисельна реалізація методу проведена в межах розгорнутого комп'ютерного експерименту, результати якого наведено у роботі. У роботі показано рисунки оптимальних температурних полів у кулі та потужностей розподілених джерел тепла у включенні, а також графіки оптимальних температур та нормальних напружень на поверхні включення залежно від геометричних параметрів та трьох різних типів зовнішнього навантаження. Усі основні результати роботи є новими. Аналітичне обґрунтування та проведені розрахунки доводять коректність та ефективність запропонованого методу.

Ключові слова: оптимальне керування, термопружний стан, потужність розподілених джерел тепла, цільовий функціонал, моделювання, узагальнений метод Фур'є, нескінченна система лінійних алгебраїчних рівнянь, параметричний розв'язок, строге обґрунтування.

Introduction. One of the important problems of modern engineering and technology is the development and use of various types of systems with internal heat sources. These include underground storage facilities, geothermal reservoirs, industrial furnaces, thermal elements of nuclear power plants, heat generators, microcircuits, and much more. Purely spherical elements with internal heat sources have found applications in biomedicine (nano-capsules with core-shell architecture for local heating of tumors), technical chemistry (localized catalysis), ecology (neutralization of organic pollutants), energy (battery components), etc. The creation of such systems is usually preceded by their mathematical modeling with subsequent optimization of the resulting models. In modeling, strength problems come to the fore, in which it is necessary to take into account the interaction of two types of fields: stress-strain and thermal. Moreover, it is the temperature stresses that are decisive for the above systems. It should be emphasized that internal heat sources significantly complicate the modeling process, in which it is necessary to take into account their power, location, and shape. Moreover, the impossibility of parametric representation of models does not allow setting optimization problems with them. Therefore, the development of effective methods for modeling the thermoelastic state of bodies with complex geometry and internal heat sources and the solution of problems of optimal control of this state using heat sources based on the constructed parametric models are very relevant.

In this paper, using the example of an elastic body with a specific geometry (a ball with an eccentric spherical inclusion), we develop a method for solving the problem of optimally controlling the body's steady-state thermoelastic state using the power of internal distributed heat sources. The method is rigorously justified. Numerical modeling and optimization results are presented, confirming the correctness and effectiveness of the proposed method.

Review of previous research results. In [1], an approach to the analysis of spherical inhomogeneities in an infinite matrix under uniaxial and triaxial loading conditions is proposed. The authors apply the decomposition method, considering the inclusion through an equivalent solution for the cavity. The key feature is the use of the principle of compatibility of deformations for statically indeterminate systems, which allows establishing a direct analytical relationship between the displacements of the poles/equator of the inhomogeneity and the contact stresses at the boundary of the inhomogeneities. The study [2] is devoted to the problem of displacement and rotation of a weakly deformed spherical rigid inclusion having the shape of a perturbed sphere. The author uses the small parameter method to describe the surface geometry and solves the problem through decomposition into separate components for arbitrary small displacements and rotations. The mathematical apparatus is based on the use of asymptotic series and expansions in tensor spherical functions, which allows satisfying the boundary conditions at the perturbed interface of media. In [3], the scale dependence of the elastic field inside a nanoscale spherical inclusion due to the influence of interfacial stresses was investigated. The authors apply *the classical Goodier approach*, which is based on the representation of solutions to *the Lamé equation in terms of volume spherical harmonics*. The use of a simplified harmonic basis allowed us to obtain explicit analytical expressions for the displacement fields, which demonstrate the emergence of non-uniformity of the elastic state at the nanoscale even under the condition of spherical symmetry of the problem. A similar approach in [4] solves the problem of the stressed state of an elastic medium with a small spherical cavity. The authors of [5] obtained a closed-form solution for the analysis of the stress distribution around rigid nanoparticles under uniaxial tension. A feature of the model is the explicit consideration of the interfacial spherical layer, the thickness of which is proportional to the inclusion radius, and the elastic characteristics differ from the matrix parameters. By approximating by the leading terms of the displacement and stress distributions, the authors were able to establish the regularities of the transformation of force fields in a three-phase system. In [6], the stress-strain state of a three-phase system consisting of a spherical inclusion and a concentric layer embedded in an infinite matrix under the action of a uniform load was investigated. The author implemented an approach based on the decomposition of the Lamé equation. Assuming a functional dependence of displacements on given deformations at infinity, a system of differential equations was obtained, the solution of which is presented in a closed analytical form. This allows us to effectively assess the influence of the geometric and mechanical parameters of the intermediate layer on the overall state of the composite.

The use of analytical and numerical methods in the study of stationary thermal and thermoelastic fields is typical for plane problems and spatial problems in canonical domains. The authors of [7] implemented a comprehensive approach to the analysis of the thermoelastic state of a semi-infinite body in the presence of stationary heat sources. The method is based on the construction *of Boussinesq functions* by reducing the problem to boundary value problems for harmonic functions in a half-space with different types of mechanical and thermal fixation of the boundary. The obtained analytical relations for displacements and stresses are interpreted as *the corresponding Green's functions*, which allows calculating the stress-strain state for arbitrary heat release in the strip domain. In [8], a method for solving the stationary thermoelasticity problem for an isotropic cylinder of finite length is proposed by decomposing the temperature field into symmetric and antisymmetric components. The authors use double expansions: into Fourier series in the axial coordinate and into *Bessel-Dini series* in the radial variable. This approach allowed reducing the boundary value problem to an infinite system of linear algebraic equations, the solution of which was found using an improved reduction method, which provides high accuracy in determining axisymmetric temperature stresses. In the study [9], a generalized method for calculating the steady-state thermoelastic state in multilayer bodies is proposed, taking into account the temperature dependence of the physical and mechanical characteristics of materials. The methodology is based on the complex application of *the Kirchhoff transformation for linearization of heat conduction equations*, *Newton's iterative algorithm*, and the apparatus of generalized functions and Green's functions. This approach allows modeling the stress-strain state in thermosensitive multilayer systems in the presence of internal heat sources. In [10], analytical solutions of mixed problems of steady-state thermal conductivity and thermoelasticity for a semi-infinite layer under the conditions of smooth contact on the end surface are presented. The research methodology is based on the decomposition of the system of Lamé equations, which allows us to isolate independent differential equations for further analysis. The application of the method of integral transformations ensured the obtaining of exact solutions that describe the interaction of thermal and mechanical fields in canonical semi-bounded regions. In the study [11], thermal stresses arising from the discrepancy between the coefficients of thermal expansion of the spherical inclusion and the matrix were analyzed. Unlike isotropic models, the author took into account the cubic symmetry of both phases of the material. To solve the problem, the method of *the equivalent Eshelby inclusion* was used, which allowed us to establish the influence of the anisotropy of elastic properties on the local concentration of stresses when the temperature changes.

In a number of studies, finite and boundary element methods and their modifications are used to solve thermoelastic problems. The authors of [12] proposed an approximate method for solving a plane stationary thermoelastic problem for a rectangular cross-section under mixed boundary conditions. The method is based on a combination of a classical solution for a uniform temperature field with the apparatus of boundary integral equations. The unknown deformation components are represented in the form of series in *Cartesian harmonics* with the addition of specific harmonic functions, which allows us to adequately model the features of the stress state at singularity points caused by a change in the type of boundary conditions. In [13], another approach is presented, namely, an effective modification of the boundary element method for solving three-dimensional thermoelasticity problems is implemented. The authors proposed an algorithm for transforming the triple integral, which describes the effect of the temperature field, into a surface integral within the general boundary integral equation. The use of appropriate Green's functions for this procedure allowed to significantly increase the computational efficiency of the method, while maintaining high accuracy of analysis for bodies with complex geometry and anisotropic properties.

Some studies investigate internal heat sources. In [14], the formation of a stationary thermal field in a solid sphere under the condition of a uniform distribution of internal heat sources was investigated. The author conducted a comparative analysis for the cases of constant thermal conductivity and its linear dependence on temperature. Due to the assumption of central symmetry, the thermal conductivity equation, which depends only on the radial coordinate, was integrated in a closed form, which allowed obtaining accurate analytical solutions for both physical models. In [15], an analytical method for solving axisymmetric problems of stationary thermoelasticity for bodies with thin thermally active inclusions was proposed. The authors used a mathematical model in which the plane inhomogeneity is replaced by an equivalent layer of heat sources given by *the Hankel integral transformation*. The solution of the problem is reduced to the analysis of *a singular integral equation with a Bessel kernel*. By using *discontinuous Weber–Schaffheitlin integrals* and expanding functions into series by *orthogonal Jacobi polynomials*, an effective analytical solution was obtained for determining the stress-strain state of the entire array. In the article [16] the authors investigated the thermoelastic state of an unbounded medium containing a single or a group of heat-generating spherical inclusions with a constant density of heat sources. The authors consider a model of thermal inclusions, where the elastic properties coincide, and only the coefficients of thermal conductivity and linear expansion differ. For a single inhomogeneity, an exact solution was obtained based on the fundamental solutions of *the Laplace equation*, while the interaction in the inclusion system was taken into account using the method of superposition of potentials centered in the geometric cells of each object. In [17], an analytical apparatus was developed for constructing Boussinesq functions in thermoelasticity problems for a half-space containing a heat-generating spherical inclusion. The authors consider a wide range of boundary conditions on the surface, from a free to a rigidly fixed boundary under different heat transfer regimes. The technique is based on a combination of thermoelastic displacement potentials with the mirror image method (symmetric or antisymmetric extension), which allowed reducing the problem for a half-space to the analysis of the interaction of two inclusions in an unbounded medium and obtaining exact solutions for a constant source density. In the study [18], the steady-state thermoelastic state of a hollow sphere made of a functional gradient material (FGM) was analyzed. A feature of the model is the consideration of all thermomechanical characteristics as continuous functions of the radial coordinate, described by power laws. Based on *the generalized Hooke's law* taking into account thermal deformations, the authors managed to obtain a closed analytical solution of the system of equations of thermal conductivity and equilibrium, which allows for a detailed study of the influence of the property gradient on the distribution of internal stresses. Similarly, in [19] the thermoelastic state of a hollow cylinder made of a functionally gradient material is defined as defined. The authors in [20] presented a mathematical model of the thermoelastic behavior of a multilayer composite hollow sphere with constant internal heat sources in each layer. The authors implemented an approach in which the temperature field is divided into a transient and a stationary component under the condition of ideal thermal contact at the phase boundaries. Based on the direct integration of the equilibrium equations in the stresses for each layer, the problem is reduced to solving a multiparameter system of linear equations. The emphasis is on the dynamics of the formation of the temperature profile, although the question of the mathematical conditionality of the obtained high-order systems remains open. In the study [21], non-stationary thermal processes in composite structures with spherical inhomogeneities during laser pulse heating were analyzed. The authors implemented a hybrid computational algorithm that combines the Eshelby equivalent inclusion method for describing local fields with the boundary element method for taking into account edge effects. Using the example of a composite disk, the significant influence of the size factor - the ratio of the sample thickness to the radius of the inclusions - on the accuracy of laser measurements of thermophysical characteristics, which is of critical importance for the metrology of inhomogeneous media, is demonstrated. In [22], a numerical-analytical model for evaluating the thermoelastic properties of composites with ellipsoidal and spherical inclusions in a cubic matrix is presented. The authors integrated the Eshelby equivalent inclusion method, which operates on the concepts of natural strains and temperature gradients, into the general scheme of the boundary element method to satisfy the boundary conditions at the boundaries of the unit cell. The formed model allowed to establish statistical regularities of the influence of the volume fraction and spatial distribution of particles on the effective macroscopic characteristics of a heterogeneous medium. In the article [23] an innovative approach to the analysis of thermomechanical characteristics of two-phase architected materials with different topological configurations is presented. The authors use *the Gilis formula to describe the topology*, and the estimation of effective elastic moduli and thermal conductivity coefficients is carried out using *the Galerkin boundary element method*. A feature of the study is the integration of the obtained data into the feature space for training deep neural networks. This allowed the use of interpreted machine learning algorithms (in particular, *Shapley values*) to identify complex dependencies between the geometric design of the phases, their volume fraction and the final physical and mechanical properties of the composite. In [24], two-dimensional steady-state thermoelasticity problems for an infinite matrix with elliptical inhomogeneity under imperfect contact conditions were solved. The model takes into account weak thermal conductivity and spring coupling between the matrix and the inhomogeneity, which is described by the corresponding interface functions. The authors determined the parameters of these functions, under which the remote heat flux and uniform heating generate linear and constant stress distributions in the inhomogeneity, respectively. The analytical expressions for the temperature and force fields obtained in a closed form allow us to evaluate the influence of defects in the interfacial coupling on the general thermomechanical state of the system. In the study [25], a hierarchical multiscale model was proposed to analyze the influence of thermal stresses on the ductile fracture of polymer nanocomposites. The material degradation process is considered through the mechanism of detachment of spherical nanoparticles from the matrix within a representative volume element containing the interfacial region. The model, verified by experimental data, allows establishing a quantitative relationship between the fracture toughness parameters and a set of thermomechanical characteristics, including the mass fraction of the filler, the coefficients of thermal expansion of the phases, as well as the geometric and stiffness parameters of the ideal contact.

Also, special attention should be paid to optimal control problems. In [26], the problem of speed of operation when heating a thermoelastic plate by internal heat sources is considered. The author formulates a mathematical model of optimal control taking into account strict constraints on both the intensity of the sources and the maximum permissible thermal stresses. The methodology is based on a combination of approaches to solving inverse problems of heat conduction with the apparatus of the finite difference method for analyzing the direct problem of thermoelasticity. This allows us to determine such a time law of energy release that minimizes the duration of heating without violating the strength criteria of the material. In the article [27], an atypical problem of optimal control of a thermoelastic body under conditions of incomplete initial data (lack of information about the initial temperature distribution and displacements) was solved. To overcome this uncertainty, the authors applied *the concept of "lossless control" (Pareto control)* developed by *J.-L. Lyons*. The methodology is based on the introduction of a sequence of cost functionals with a regularizing parameter. It is proved that the limit transition, when this parameter tends to zero, allows transforming the strategy of minimum loss into a full-fledged lossless control, which ensures stable control of deformations even with a deficit of initial information. In [28], based on the apparatus of inverse problems of thermomechanics, a mathematical statement of the problem of optimal heating action speed of thermosensitive bodies of canonical form (layer, cylinder, sphere) was formulated. A feature of the model is the consideration of the dependence of the physical and mechanical properties of the material on temperature and the presence of restrictions on the intensity of control and the limiting tangential stresses in the zone of plastic deformations. The authors developed a stable numerical algorithm for constructing the optimal heating law, which allows minimizing the process time while strictly observing the strength criteria under conditions of inelastic behavior of the material. In the study [29], a method of quasi-static inverse thermoelasticity problem was developed to optimize the heating rate of a long hollow cylinder under conditions of non-stationary non-axisymmetric thermal regime. The authors proposed a control algorithm that takes into account the angular dependence of the temperature field and strict constraints on thermoelastic stresses. The mathematical model was reduced to the Fredholm integral equation of the first kind, for which a stable regularized solution method was developed. This allowed obtaining optimal time and space heating parameters that minimize the duration of transient processes without violating the strength criteria of the structure. In [30], a mathematical formulation was formulated and an algorithm was developed for numerically solving a two-dimensional problem of optimal speed-dependent heating control of a long rectangular parallelepiped. The study is based on a model of elastic-plastic deformation of a thermosensitive material, the properties of which depend on temperature. The authors implemented a strategy for minimizing the heating time while observing restrictions on the intensity of thermal influence and the maximum permissible intensity of tangential stresses in the plasticity zone. This allows optimizing the heat treatment of elements with a rectangular cross-section, preventing their premature failure or undesirable residual deformations. In [31], the problem of determining the optimal temperature regime for heating a piecewise homogeneous cylindrical glass shell under the condition of minimizing meridional and circular normal stresses was investigated. Heating is carried out convectively using external heat sources with a thermally insulated inner surface. The authors proposed a method of step-by-step parametric optimization that allows finding such a law of change in the temperature of the external environment that ensures the achievement of a given thermal state at a fixed point in time while strictly observing the restrictions on the heating rate and the level of thermal stresses. Thanks to the procedure of averaging the characteristics over the shell thickness, the mathematical model was reduced to one-dimensional in spatial coordinates, which significantly simplifies the computational implementation of the algorithm. In [32], a method for optimal control of the axisymmetric thermally stressed state of a solid cylinder by modifying the spatial distribution of volumetric heat sources is presented. The methodology is based on the application of the variational method of homogeneous solutions, which allows for effective minimization of thermal stresses in closed regions. In [33], a theoretical justification and a numerical algorithm for solving the coupled optimal control problem under conditions of stationary thermoelasticity are presented. The author uses the apparatus of classical variational calculus and *the method of Lagrange* multipliers to derive the necessary optimality conditions that form a system of conjugate differential equations for the temperature field and the displacement field. The practical implementation of the method is based on a combination of the finite element method for spatial discretization and the iterative conjugate gradient method for minimizing the objective functional, which allows for effective optimization of the boundary control for bodies of arbitrary finite shape. In the article [34] proposes a method for determining the optimal stress control of an elastic space containing a coaxial cavity and inclusion using the temperature distribution on the cavity surface. The problem is reduced to an equivalent problem of the conditional extremum of a quadratic functional. A method for parametrically solving infinite systems of linear algebraic equations is also proposed.

The above bibliographic review of the latest research in the field of thermoelasticity and optimal stress control using a temperature field allows us to draw the following conclusions:

- aside from the work of the authors [34], there are virtually no studies of optimal control problems for the thermoelastic state of multiphase bodies with complex geometries;
- the method proposed in [34] for solving problems of optimal control of the thermoelastic state of a body using a temperature field requires further development not only for bodies with different geometries but also for other types of physical control;
- the method for solving the equivalent problem proposed in [34] needs to be expanded to a broader class of controls, which requires a more detailed justification.

Thus, the aim of this work is to further develop the method for optimal control of the thermoelastic state of a composite body using the power of internal distributed heat sources, to verify the correctness and efficiency of the method, and to apply it to the optimal modeling of the thermoelastic state of a ball with an eccentric spherical inclusion.

Mathematical formulation of the problem. Consider an elastic ball centered at a point O_1 of radius R_1, which has an eccentric spherical elastic inclusion $B_{R_2}(O_2)$ centered at a point O_2 of radius R_2 ($|O_1O_2| + R_2 < R_1$), made of another material. Let us introduce two spherical coordinate systems (r_j, θ_j, φ) $(j = 1, 2)$, the origins of which are aligned with the points O_j ($|\overrightarrow{O_1O_2}| = z_{12}$), and their axes of symmetry have the direction of the vector $\overrightarrow{O_1O_2}$. Let us denote the domains $\Omega_1 = B_{R_1}(O_1) \setminus \overline{B_{R_2}(O_2)}$, $\Omega_2 = B_{R_2}(O_2)$. We will assume that the material of the part of the sphere that occupies the domain has thermomechanical characteristics $(G_j, \nu_j, \alpha_j, k_j)$ $(j = 1, 2)$, where G is the shear modulus, ν is *the Poisson's ratio*, α is the coefficient of linear thermal expansion, k is the coefficient of thermal conductivity.

At the first stage, we will solve the direct problem of constructing a parametric model of the thermoelastic steady-state of a ball $B_{R_1}(O_1)$, whose surface Γ_1 is under uniform pressure and has a temperature T_0. We will assume that the conditions of ideal thermomechanical contact are met on the surface of the inclusion Γ_2, and the inclusion itself releases heat with a density $k_2 g(r_2, \theta_2)$. Note that the boundary conditions on the surface may be different, and those chosen in the article do not limit the generality of the model that will be constructed.

The mathematical model of the thermoelastic state of the system under consideration is a boundary value problem for a system of elliptic partial differential equations $(j = 1, 2)$

$$\vec{\nabla}^2 \vec{U}_j + \frac{1}{1-2\nu_j} \vec{\nabla}\left(\vec{\nabla} \vec{U}_j\right) = \alpha_j \frac{2+2\nu_j}{1-2\nu_j} \vec{\nabla} T_j, \ \vec{x} \in \Omega_j; \tag{1}$$

$$\vec{\nabla}^2 T_j + \delta_{j,2} g = 0, \ \vec{x} \in \Omega_j \tag{2}$$

with boundary conditions

$$\left(T_1\right)_{|\Gamma_1} = T_0, \tag{3}$$

$$\left(F\vec{U}_1\right)_{|\Gamma_1} = 2G_1 \sum_{n=0}^{\infty} \left[f_n^{(1)} P_n(\cos\theta_1) \vec{e}_{r_1} + f_n^{(2)} P_n^1(\cos\theta_1) \vec{e}_{\theta_1} \right] \tag{4}$$

and the conditions of conjugation of thermomechanical fields

$$\left(T_1\right)_{|\Gamma_2} = \left(T_2\right)_{|\Gamma_2}, \ \left(k_1 \frac{\partial T_1}{\partial n_2}\right)_{|\Gamma_2} = \left(k_2 \frac{\partial T_2}{\partial n_2}\right)_{|\Gamma_2}; \tag{5}$$

$$\left(\vec{U}_1\right)_{|\Gamma_2} = \left(\vec{U}_2\right)_{|\Gamma_2}, \ \left(F\vec{U}_1\right)_{|\Gamma_2} = \left(F\vec{U}_2\right)_{|\Gamma_2}. \tag{6}$$

Here T_j, $\vec{U}_j$ $(j = 1, 2)$ denotes the temperature field and the displacement field in the domain Ω_j, $F\vec{U}_j$ is the stress vector on the surface Γ_j with the normal $\vec{n}_1 = \vec{e}_{r_j}$, corresponding to the displacement vector $\vec{U}_j$, $\delta_{j,k}$ is Kronecker delta symbol, $\vec{\nabla}$ is the operator of the nabla, $\{\vec{e}_{r_j}, \vec{e}_{\theta_j}\}$ are the unit vectors of the spherical coordinate system with the origin at the point O_j, $\vec{x}$ is the point of three-dimensional space, the Cartesian coordinates of which are related to the spherical coordinates by the formulas

$$x = r_1 \sin\theta_1 \cos\varphi_1 = r_2 \sin\theta_2 \cos\varphi_2, \;\; y = r_1 \sin\theta_1 \sin\varphi_1 = r_2 \sin\theta_2 \sin\varphi_2, \;\; z = r_1 \cos\theta_1 = z_{12} + r_2 \cos\theta_2 .$$

At the second stage, using the constructed model, the problem of optimal control of the thermoelastic state using the power of heat sources is solved. The optimal control problem is posed for the objective functional

$$\frac{1}{|\Gamma_2|} \int_{\Gamma_2} \left| F\vec{U}_2 \right|^2 ds \to \min \tag{7}$$

with a constraint on the mean square of the reduced power of distributed heat sources

$$\frac{1}{|\Omega_2|} \int_{\Omega_2} g^2 (r_2, \theta_2) d\vec{x} = D^2 . \tag{8}$$

Regarding the function $g(r_2, \theta_2)$, we will assume that it is harmonic in the domain Ω_2, in the closure of which it is represented by an absolutely and uniformly convergent series

$$g(r_2, \theta_2) = \sum_{n=0}^{\infty} g_n \left(\frac{r_2}{R_2} \right)^n P_n(\cos\theta_2), \tag{9}$$

where $P_n(x)$ is the Legendre polynomial.

Modeling of temperature and thermoelastic fields. At the first stage, we will solve the direct problem of modeling temperature and thermoelastic fields in the domains Ω_j, assuming that function (9) is given. The solution to problem (2), (3), (5), (9) is constructed in the form

$$T_1(\vec{x}) = \sum_{n=0}^{\infty} t_n^{(1)} R_1^{-n} w_n^- (r_1, \theta_1) + \sum_{n=0}^{\infty} t_n^{(2)} R_2^{n+1} w_n^+ (r_2, \theta_2), \quad \vec{x} \in \Omega_1 ; \tag{10}$$

$$T_2(\vec{x}) = \sum_{n=0}^{\infty} R_2^{-n} \left[d_n r_2^2 + c_n \right] w_n^- (r_2, \theta_2), \;\; \vec{x} \in \Omega_2 \tag{11}$$

with unknown parameters $\left\{ t_n^{(j)} \right\}_{n=0, j=1}^{\infty, 2}$, $\{d_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$. Here the axisymmetric basis solutions of the Laplace equation for the exterior $\Omega^+ = \{r > R\}$ and interior $\Omega^- = \{r < R\}$ of the sphere are denoted by

$$w_n^{\pm}(r, \theta) = r^{\mp(n+1/2)-1/2} P_n(\cos\theta), \tag{12}$$

where the sign $+(-)$ corresponds to the external (internal) solution.

Let us find the parameters d_n in solution (9) of Poisson's equation (2), for which we apply the Laplace operator to the function $T_2(r_2, \theta_2)$

$$\Delta T_2(r_2, \theta_2) = \sum_{n=0}^{\infty} R_2^{-n} 2(2n+3) d_n w_n^- (r_2, \theta_2).$$

Comparing this result with (2) and (9), we obtain

$$d_n = -\frac{g_n}{2(2n+3)} . \tag{13}$$

Let's use the formulas [35] ($n = 0 \div \infty$)

$$w_n^+ (r_2, \theta_2) = \sum_{k=n}^{\infty} C_k^n z_{12}^{k-n} w_k^+ (r_1, \theta_1), \;\; r_1 > z_{12} ; \tag{14}$$

$$w_n^- (r_1, \theta_1) = \sum_{k=0}^{n} C_n^k z_{12}^{n-k} w_k^- (r_2, \theta_2), \tag{15}$$

where C_n^k is the binomial coefficient, to write the temperature field $T_1(\vec{x})$ in coordinate systems with origins at points O_j

$$T_1(\vec{x}) = \sum_{n=0}^{\infty} t_n^{(1)} R_1^{-n} w_n^-(r_1,\theta_1) + \sum_{n=0}^{\infty} w_n^+(r_1,\theta_1) \sum_{k=0}^{n} C_n^k R_2^{k+1} z_{12}^{n-k} t_k^{(2)}, \tag{16}$$

$$T_1(\vec{x}) = \sum_{n=0}^{\infty} t_n^{(2)} R_2^{n+1} w_n^+(r_2,\theta_2) + \sum_{n=0}^{\infty} w_n^-(r_2,\theta_2) \sum_{k=n}^{\infty} C_k^n R_1^{-k} z_{12}^{k-n} t_k^{(1)}. \tag{17}$$

Satisfying the conditions of conjugation of thermal fields on the surface Γ_2 of the inclusion and the boundary condition on the surface Γ_1, we arrive at an infinite system of linear algebraic equations with respect to the unknowns $\left\{t_n^{(j)}\right\}_{n=0,\,j=1}^{\infty,2}$, $\left\{c_n\right\}_{n=0}^{\infty}$

$$t_n^{(1)} + \sum_{k=0}^{n} u_{n,k}^+ t_k^{(2)} = T_0 \delta_{n,0}, \; n = 0 \div \infty; \tag{18}$$

$$t_n^{(2)} + K_n^{(1)} \sum_{k=n}^{\infty} u_{n,k}^- t_k^{(1)} = K_n^{(2)} \frac{R_2^2 g_n}{2n+3}, \; n = 0 \div \infty; \tag{19}$$

$$c_n = t_n^{(2)} + \sum_{k=n}^{\infty} u_{n,k}^- t_k^{(1)} + \frac{R_2^2 g_n}{2(2n+3)}, \; n = 0 \div \infty, \tag{20}$$

where

$$u_{n,k}^+ = C_n^k \omega_{n+1,k+1}, \; u_{n,k}^- = C_k^n \omega_{k,n}, \; \omega_{n,k} = \left(\frac{z_{12}}{R_1}\right)^n \left(\frac{R_2}{z_{12}}\right)^k, \; K_n^{(1)} = \frac{(k_2 - k_1)n}{k_2 n + k_1(n+1)}, \; K_n^{(2)} = \frac{k_2}{k_2 n + k_1(n+1)}.$$

Theorem 1. *The operator of the system (18), (19) is a Fredholm operator in the space* l_2^2 *under the condition* $z_{12} + R_2 < R_1$.

Proof. To prove the theorem, it is sufficient to show the convergence of the series

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} C_n^k \omega_{n,k} = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} C_k^n \omega_{k,n}.$$

The latter follows from the condition of the theorem, since

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} C_n^k \omega_{n,k} = \sum_{n=0}^{\infty} \left(\frac{z_{12}}{R_1}\right)^n \sum_{k=0}^{n} C_n^k \left(\frac{R_2}{z_{12}}\right)^k = \sum_{n=0}^{\infty} \left(\frac{z_{12}}{R_1}\right)^n \left(1 + \frac{R_2}{z_{12}}\right)^n$$

and the last series converges as the sum of a geometric sequence with a denominator less than unity.

Now we will solve problem (1), (4), (6) for the temperature field, which is given by formulas (10), (11), (18) – (20). For this purpose, we will use the generalized Fourier method. In the article [36], axisymmetric sets of vector partial solutions of the Lamé equation

$$\left\{\vec{W}_{1,0}^+(r,\theta), \vec{W}_{1,n}^+(r,\theta), \vec{W}_{2,n}^+(r,\theta)\right\}_{n=1}^{\infty} \quad \left(\left\{\vec{W}_{2,0}^-(r,\theta), \vec{W}_{1,n}^-(r,\theta), \vec{W}_{2,n}^-(r,\theta)\right\}_{n=1}^{\infty}\right)$$

for the exterior Ω^+ (interior Ω^-) of the sphere were introduced and it was proved that they form systems of basis solutions in the corresponding domains $\Omega^\pm$. Here

$$\vec{W}_{1,n}^\pm(r,\theta) = \vec{\nabla} w_n^\pm(r,\theta), \tag{21}$$

$$\vec{W}_{2,n}^\pm(r,\theta) = \chi_n^\pm \vec{V}_n^\pm(r,\theta) + \zeta_n^\pm \vec{W}_{1,n}^\pm(r,\theta), \tag{22}$$

where

$$\vec{V}_n^\pm(r,\theta) = \vec{\nabla}\left[r^2 w_n^\pm(r,\theta)\right], \tag{23}$$

$$\chi_n^+ = (4\nu - 3)n + 2\nu - 2, \; \zeta_n^+ = (2n-1)(2\nu - 2), \; \chi_n^- = (4\nu - 3)n + 2\nu - 1, \; \zeta_n^- = (2n+3)(2\nu - 2). \tag{24}$$

In addition, we introduce the following vector functions

$$\vec{V}_n^{-(1)}(r,\theta) = \vec{\nabla}\left[r^4 w^-(r,\theta)\right]. \tag{25}$$

The authors obtained the following result.

Theorem 2. *For vector functions (21) – (23) in spherical coordinate systems with origins at points* O_j, *the addition theorems hold*

$$\vec{W}_{1,n}^+(r_2,\theta_2) = \sum_{k=n}^{\infty} C_k^n z_{12}^{k-n} \vec{W}_{1,k}^+(r_1,\theta_1), \; r_1 > z_{12}, \; n = 0 \div \infty; \tag{26}$$

$$\vec{W}^{+}_{2,n}\left(r_2,\theta_2\right)=\sum_{k=n}^{\infty}C^{n-1}_{k-1}\gamma^{+(2)}_{n,k}z_{12}^{k-n}\vec{W}^{+}_{2,k}\left(r_1,\theta_1\right)+\sum_{k=n-1}^{\infty}C^{n-1}_{k}\gamma^{+(1)}_{n,k}z_{12}^{k-n+2}\vec{W}^{+}_{1,k}\left(r_1,\theta_1\right),\ r_1>z_{12},\ n=1\div\infty\,; \tag{27}$$

$$\vec{W}^{-}_{1,n}\left(r_1,\theta_1\right)=\sum_{k=0}^{n}C^{k}_{n}z_{12}^{n-k}\vec{W}^{-}_{1,k}\left(r_2,\theta_2\right),\ n=1\div\infty\,; \tag{28}$$

$$\vec{W}^{-}_{2,n}\left(r_1,\theta_1\right)=\sum_{k=0}^{n}C^{k+1}_{n+1}\gamma^{-(2)}_{n,k}z_{12}^{n-k}\vec{W}^{-}_{2,k}\left(r_2,\theta_2\right)+\sum_{k=0}^{n+1}C^{k}_{n+1}\gamma^{-(1)}_{n,k}z_{12}^{n-k+2}\vec{W}^{-}_{1,k}\left(r_2,\theta_2\right),\ n=0\div\infty\,; \tag{29}$$

$$\vec{V}^{+}_{n}\left(r_2,\theta_2\right)=\sum_{k=n}^{\infty}C^{n}_{k}\gamma^{+(2)}_{n,k}z_{12}^{k-n}\vec{V}^{+}_{k}\left(r_1,\theta_1\right)+\sum_{k=n-1}^{\infty}C^{n}_{k+1}\lambda^{+(1)}_{n,k}z_{12}^{k-n+2}\vec{W}^{+}_{1,k}\left(r_1,\theta_1\right),\ r_1>z_{12},\ n=1\div\infty\,; \tag{30}$$

$$\vec{V}^{-}_{n}\left(r_1,\theta_1\right)=\sum_{k=0}^{n}C^{k}_{n}\gamma^{-(2)}_{n,k}z_{12}^{n-k}\vec{V}^{-}_{k}\left(r_2,\theta_2\right)+\sum_{k=0}^{n+1}C^{k}_{n+1}\lambda^{-(1)}_{n,k}z_{12}^{n-k+2}\vec{W}^{-}_{1,k}\left(r_2,\theta_2\right),\ n=0\div\infty\,; \tag{31}$$

where

$$\gamma^{-(1)}_{n,k}=\frac{k-2nk-3n+5-4\nu_1}{2k+3},\ \gamma^{+(2)}_{n,k}=\frac{2n-1}{2k-1},\ \gamma^{-(1)}_{n,k}=\frac{2nk+3k-n-5+4\nu_1}{2k-1},\ \gamma^{-(2)}_{n,k}=\frac{2n+3}{2k+3},$$

$$\lambda^{+(1)}_{n,k}=\frac{k+1-2nk-3n}{\left(k+1\right)\left(2k+3\right)},\ \lambda^{-(1)}_{n,k}=\frac{2nk+3k-n-1}{\left(n+1\right)\left(2k-1\right)}.$$

As is known, the general solution of the inhomogeneous equation (1) in the domain Ω_j $\left(j=1\div2\right)$ can be written in the following form:

$$\vec{U}_j\left(\vec{x}\right)=\vec{U}^G_j\left(\vec{x}\right)+\vec{U}^T_j\left(\vec{x}\right), \tag{32}$$

where $\vec{U}^G_j\left(\vec{x}\right)$ is a general solution of the corresponding homogeneous equation, $\vec{U}^T_j\left(\vec{x}\right)$ is a partial solution of a non-homogeneous equation (thermal displacements). Due to the basis nature of solutions (21), (22), the general solution of the homogeneous equation (1) in the domains $\left\{\Omega_j\right\}_{j=1}^{2}$ can be written as follows:

$$\vec{U}^G_1\left(\vec{x}\right)=\sum_{n=1}^{\infty}a^{(1)}_{1,n}R_1^{-n+2}\vec{W}^{-}_{1,n}\left(r_1,\theta_1\right)+\sum_{n=0}^{\infty}a^{(1)}_{2,n}R_1^{-n}\vec{W}^{-}_{2,n}\left(r_1,\theta_1\right)+\sum_{n=0}^{\infty}a^{(2)}_{1,n}R_2^{n+3}\vec{W}^{+}_{1,n}\left(r_2,\theta_2\right)+$$
$$+\sum_{n=1}^{\infty}a^{(2)}_{2,n}R_2^{n+1}\vec{W}^{+}_{2,n}\left(r_2,\theta_2\right),\ \vec{x}\in\Omega_1\ ; \tag{33}$$

$$\vec{U}^G_2\left(\vec{x}\right)=\sum_{n=1}^{\infty}b_{1,n}R_2^{-n}\vec{U}^G_2\left(\vec{x}\right)=\sum_{n=1}^{\infty}b_{1,n}R_2^{-n+2}\vec{W}^{-}_{1,n}\left(r_2,\theta_2\right)+\sum_{n=0}^{\infty}b_{2,n}R_2^{-n}\vec{W}^{-}_{2,n}\left(r_2,\theta_2\right),\ \vec{x}\in\Omega_2\,. \tag{34}$$

Here $a^{(j)}_{i,n}$, $b_{i,n}$ are the unknown parameters of the model.

We will look for a partial solution $\vec{U}^T_j\left(\vec{x}\right)$ in the domain Ω_j in the form

$$\vec{U}^T_j\left(\vec{x}\right)=\vec{\nabla}\Phi_j\left(\vec{x}\right).$$

Then we obtain the Poisson equation for the function $\Phi_j\left(\vec{x}\right)$

$$\Delta\Phi_j\left(\vec{x}\right)=2\tilde{\alpha}_jT_j\left(\vec{x}\right),\ \vec{x}\in\Omega_j,\ \tilde{\alpha}_j=\frac{\alpha_j}{2}\frac{1+\nu_j}{1-\nu_j},$$

whose solution can be written as:

$$\Phi_1\left(\vec{x}\right)=\tilde{\alpha}_1\sum_{n=0}^{\infty}\frac{R_1^{-n}t^{(1)}_n}{2n+3}r_1^2w^{-}_n\left(r_1,\theta_1\right)-\tilde{\alpha}_1\sum_{n=0}^{\infty}\frac{R_2^{n+1}t^{(2)}_n}{2n-1}r_1^2w^{+}_n\left(r_2,\theta_2\right),$$

$$\Phi_2\left(\vec{x}\right)=\tilde{\alpha}_2\sum_{n=0}^{\infty}\frac{R_2^{-n}}{2n+3}\left[r_2^2c_n-\frac{r_2^4g_n}{4\left(2n+5\right)}\right]w^{-}_n\left(r_2,\theta_2\right).$$

Thus, the vectors of thermal displacements in the domains $\left(\Omega_i\right)_{i=1}^{2}$ are given by the formulas

$$\vec{U}^T_1\left(\vec{x}\right)=\tilde{\alpha}_1\sum_{n=0}^{\infty}\frac{R_1^{-n}t^{(1)}_n}{2n+3}\vec{V}^{-}_n\left(r_1,\theta_1\right)-\tilde{\alpha}_1\sum_{n=0}^{\infty}\frac{R_2^{n+1}t^{(2)}_n}{2n-1}\vec{V}^{+}_n\left(r_2,\theta_2\right),\ \vec{x}\in\Omega_1\ ; \tag{35}$$

$$\vec{U}^T_2\left(\vec{x}\right)=-\tilde{\alpha}_2\sum_{n=0}^{\infty}\frac{R_2^{-n}g_n}{4\left(2n+5\right)\left(2n+3\right)}\vec{V}^{-(1)}_n\left(r_2,\theta_2\right)+\tilde{\alpha}_2\sum_{n=0}^{\infty}\frac{R_2^{-n}c_n}{2n+3}\vec{V}^{-}_n\left(r_2,\theta_2\right),\ \vec{x}\in\Omega_2\,. \tag{36}$$

Formulas (26) – (31) allow us to write a vector function $\vec{U}_1(\vec{x})$ in a spherical coordinate system with the origin at the point O_j

$$\vec{U}_1(r_1,\theta_1)=\sum_{n=0}^{\infty}\vec{W}_{1,n}^{+}(r_1,\theta_1)\sum_{k=0}^{n}C_n^k z_{12}^{n-k}R_2^{k+3}a_{1,k}^{(2)}+\sum_{n=1}^{\infty}a_{1,k}^{(1)}R_1^{-n+2}\vec{W}_{1,n}^{-}(r_1,\theta_1)+\sum_{n=0}^{\infty}a_{2,n}^{(1)}R_1^{-n}\vec{W}_{2,n}^{-}(r_1,\theta_1)+$$

$$+\sum_{n=1}^{\infty}\vec{W}_{2,n}^{+}(r_1,\theta_1)\sum_{k=1}^{n}C_{n-1}^{k-1}\gamma_{k,n}^{+(2)}z_{12}^{n-k}R_2^{k+1}a_{2,k}^{(2)}+\sum_{n=0}^{\infty}\vec{W}_{1,n}^{+}(r_1,\theta_1)\sum_{k=1}^{n+1}C_n^{k-1}\gamma_{k,n}^{+(1)}z_{12}^{n-k+2}R_2^{k+1}a_{2,k}^{(2)}\tilde{\alpha}_1\sum_{n=0}^{\infty}\frac{t_n^{(1)}}{2n+3}R_1^{-n}\vec{V}_n^{-}(r_1,\theta_1)-$$

$$-\tilde{\alpha}_1\sum_{n=0}^{\infty}\vec{V}_n^{+}(r_1,\theta_1)\sum_{k=0}^{n}C_n^k\gamma_{k,n}^{+(2)}z_{12}^{n-k}\frac{R_2^{k+1}t_k^{(2)}}{2k-1}-\tilde{\alpha}_1\sum_{n=0}^{\infty}\vec{W}_{1,n}^{+}(r_1,\theta_1)\sum_{k=1}^{n+1}C_{n+1}^k\lambda_{k,n}^{+(1)}z_{12}^{n-k+2}\frac{R_2^{k+1}t_k^{(2)}}{2k-1}, \tag{37}$$

$$\vec{U}_1(r_2,\theta_2)=\sum_{n=1}^{\infty}\vec{W}_{1,n}^{-}(r_2,\theta_2)\sum_{k=n}^{\infty}C_k^n z_{12}^{k-n}R_1^{-k+2}a_{1,k}^{(1)}+\sum_{n=0}^{\infty}a_{1,n}^{(2)}R_2^{n+3}\vec{W}_{1,n}^{+}(r_2,\theta_2)+\sum_{n=1}^{\infty}a_{2,n}^{(2)}R_2^{n+1}\vec{W}_{2,n}^{+}(r_2,\theta_2)+$$

$$+\sum_{n=0}^{\infty}\vec{W}_{2,n}^{-}(r_2,\theta_2)\sum_{k=n}^{\infty}C_{k+1}^{n+1}\gamma_{k,n}^{-(2)}z_{12}^{k-n}R_1^{-k}a_{2,k}^{(1)}+\sum_{n=1}^{\infty}\vec{W}_{1,n}^{-}(r_2,\theta_2)\sum_{k=n-1}^{\infty}C_{k+1}^{n}\gamma_{k,n}^{-(1)}z_{12}^{k-n+2}R_1^{-k}a_{2,k}^{(1)}-\tilde{\alpha}_1\sum_{n=0}^{\infty}\frac{t_n^{(2)}}{2n-1}R_2^{n+1}\vec{V}_n^{+}(r_2,\theta_2)+$$

$$+\tilde{\alpha}_1\sum_{n=0}^{\infty}\vec{V}_n^{-}(r_2,\theta_2)\sum_{k=n}^{\infty}C_k^n\gamma_{k,n}^{-(2)}z_{12}^{k-n}\frac{R_1^{-k}t_k^{(1)}}{2k+3}+\tilde{\alpha}_1\sum_{n=1}^{\infty}\vec{W}_{1,n}^{-}(r_2,\theta_2)\sum_{k=n-1}^{\infty}C_{k+1}^n\lambda_{k,n}^{-(1)}z_{12}^{k-n+2}\frac{R_1^{-k}t_k^{(1)}}{2k+3}. \tag{38}$$

According to formulas (34), (36)

$$\vec{U}_2(r_2,\theta_2)=\sum_{n=1}^{\infty}b_{1,n}R_2^{-n+2}\vec{W}_{1,n}^{-}(r_2,\theta_2)+\sum_{n=0}^{\infty}b_{2,n}R_2^{-n}\vec{W}_{2,n}^{-}(r_2,\theta_2)+\tilde{\alpha}_2\sum_{n=0}^{\infty}\frac{R_2^{-n}c_n}{2n+3}\vec{V}_n^{-}(r_2,\theta_2)-$$

$$-\tilde{\alpha}_2\sum_{n=0}^{\infty}\frac{R_2^{-n}g_n}{4(2n+3)(2n+5)}\vec{V}_n^{-(1)}(r_2,\theta_2). \tag{39}$$

Let us pass in formulas (36) – (38) from displacements to stresses on surfaces Γ_1 and Γ_2 with normal vectors $\vec{n}_1=\vec{e}_{r_1}$ and $\vec{n}_2=\vec{e}_{r_2}$ respectively and write them in coordinate form. After satisfying boundary condition (4) and conjugation conditions (6), we obtain an infinite system of linear algebraic equations with respect to unknown

$$n(n-1)a_{1,n}^{(1)}+\rho_{1,n}^{-(1)}a_{2,n}^{(1)}+(n+1)(n+2)\sum_{k=0}^{n}u_{n,k}^{+(1)}a_{1,k}^{(2)}+\rho_{1,n}^{+(1)}\sum_{k=1}^{n}u_{n,k}^{+(2)}a_{2,k}^{(2)}+(n+1)(n+2)\sum_{k=1}^{n+1}u_{n,k}^{+(3)}a_{2,k}^{(2)}+$$

$$+\tilde{\alpha}_1\left[\frac{(n+1)(n+2)}{2n+3}-2\right]t_n^{(1)}-\tilde{\alpha}_1\left[n(n-1)+4n-2\right]\sum_{k=0}^{n}u_{n,k}^{+(4)}\frac{t_k^{(2)}}{2k-1}-$$

$$-\tilde{\alpha}_1(n+1)(n+2)\sum_{k=1}^{n+1}u_{n,k}^{+(5)}\frac{t_k^{(2)}}{2k-1}=f_n^{(1)},\ n=0\div\infty; \tag{40}$$

$$(n-1)a_{1,n}^{(1)}+\rho_{2,n}^{-(1)}a_{2,n}^{(1)}-(n+2)\sum_{k=0}^{n}u_{n,k}^{+(1)}a_{1,k}^{(2)}+\rho_{2,n}^{+(1)}\sum_{k=1}^{n}u_{n,k}^{+(2)}a_{2,k}^{(2)}-(n+2)\sum_{k=1}^{n+1}u_{n,k}^{+(3)}a_{2,k}^{(2)}+\tilde{\alpha}_1\frac{(n+1)t_n^{(1)}}{2n+3}+$$

$$+\tilde{\alpha}_1 n\sum_{k=0}^{n}u_{n,k}^{+(4)}\frac{t_k^{(2)}}{2k-1}+\tilde{\alpha}_1(n+2)\sum_{k=1}^{n+1}u_{n,k}^{+(5)}\frac{t_k^{(2)}}{2k-1}=f_n^{(2)},\ n=1\div\infty; \tag{41}$$

$$-(n+1)a_{1,n}^{(2)}+\beta_{1,n}^{+(1)}a_{2,n}^{(2)}+n\sum_{k=n}^{\infty}u_{n,k}^{-(1)}a_{1,k}^{(1)}+\beta_{1,n}^{-(1)}\sum_{k=n}^{\infty}u_{n,k}^{-(2)}a_{2,k}^{(1)}+n\sum_{k=n-1}^{\infty}u_{n,k}^{-(3)}a_{2,k}^{(1)}+\tilde{\alpha}_1\frac{(n-1)t_n^{(2)}}{2n-1}+\tilde{\alpha}_1(n+2)\sum_{k=n}^{\infty}u_{n,k}^{-(4)}\frac{t_k^{(1)}}{2k+3}+$$

$$+\tilde{\alpha}_1 n\sum_{k=n-1}^{\infty}u_{n,k}^{-(5)}\frac{t_k^{(1)}}{2k+3}=nb_{1,n}+\beta_{1,n}^{-(2)}b_{2,n}-\tilde{\alpha}_2\frac{(n+4)g_nR_2^2}{4(2n+3)(2n+5)}+\tilde{\alpha}_2\frac{(n+2)c_n}{2n+3},\ n=0\div\infty; \tag{42}$$

$$a_{1,n}^{(2)}+\beta_{2,n}^{+(1)}a_{2,n}^{(2)}+\sum_{k=n}^{\infty}u_{n,k}^{-(1)}a_{1,k}^{(1)}+\beta_{2,n}^{-(1)}\sum_{k=n}^{\infty}u_{n,k}^{-(2)}a_{2,k}^{(1)}+\sum_{k=n-1}^{\infty}u_{n,k}^{-(3)}a_{2,k}^{(1)}-\tilde{\alpha}_1\frac{t_n^{(2)}}{2n-1}+\tilde{\alpha}_1\sum_{k=n}^{\infty}u_{n,k}^{-(4)}\frac{t_k^{(1)}}{2k+3}+$$

$$+\tilde{\alpha}_1\sum_{k=n-1}^{\infty}u_{n,k}^{-(5)}\frac{t_k^{(1)}}{2k+3}=b_{1,n}+\beta_{2,n}^{-(2)}b_{2,n}-\tilde{\alpha}_2\frac{g_nR_2^2}{4(2n+3)(2n+5)}+\tilde{\alpha}_2\frac{c_n}{2n+3},\ n=1\div\infty; \tag{43}$$

$$(n+2)(n+1)a_{1,n}^{(2)}+\rho_{1,n}^{+(1)}a_{2,n}^{(2)}+n(n-1)\sum_{k=n}^{\infty}u_{n,k}^{-(1)}a_{1,k}^{(1)}+\rho_{1,n}^{-(1)}\sum_{k=n}^{\infty}u_{n,k}^{-(2)}a_{2,k}^{(1)}+n(n-1)\sum_{k=n-1}^{\infty}u_{n,k}^{-(3)}a_{2,k}^{(1)}-$$

$$-\tilde{\alpha}_1\frac{\left[n(n-1)+4n-2\right]t_n^{(2)}}{2n-1}+\tilde{\alpha}_1\left[(n+1)(n+2)-4n-6\right]\sum_{k=n}^{\infty}u_{n,k}^{-(4)}\frac{t_k^{(1)}}{2k+3}+\tilde{\alpha}_1 n(n-1)\sum_{k=n-1}^{\infty}u_{n,k}^{-(5)}\frac{t_k^{(1)}}{2k+3}=$$

$$=\frac{G_2}{G_1}\left\{n(n-1)b_{1,n}+\rho_{1,n}^{-(2)}b_{2,n}-\tilde{\alpha}_2\frac{\left[(n+3)(n+4)-4(2n+5)\right]g_nR_2^2}{4(2n+3)(2n+5)}+\right.$$

$$\left.+\tilde{\alpha}_2\frac{\left[(n+1)(n+2)-2(2n+3)\right]c_n}{2n+3}\right\},\ n=0\div\infty\,; \tag{44}$$

$$-(n+2)a_{1,n}^{(2)}+\rho_{2,n}^{+(1)}a_{2,n}^{(2)}+(n-1)\sum_{k=n}^{\infty}u_{n,k}^{-(1)}a_{1,k}^{(1)}+\rho_{2,n}^{-(1)}\sum_{k=n}^{\infty}u_{n,k}^{-(2)}a_{2,k}^{(1)}+(n-1)\sum_{k=n-1}^{\infty}u_{n,k}^{-(3)}a_{2,k}^{(1)}+\tilde{\alpha}_1\frac{nt_n^{(2)}}{2n-1}+$$

$$+\tilde{\alpha}_1(n+1)\sum_{k=n}^{\infty}u_{n,k}^{-(4)}\frac{t_k^{(1)}}{2k+3}+\tilde{\alpha}_1(n-1)\sum_{k=n-1}^{\infty}u_{n,k}^{-(5)}\frac{t_k^{(1)}}{2k+3}=$$

$$\frac{G_2}{G_1}\left\{(n-1)b_{1,n}+\rho_{2,n}^{-(2)}b_{2,n}-\tilde{\alpha}_2\frac{(n+3)g_nR_2^2}{4(2n+3)(2n+5)}+\tilde{\alpha}_2\frac{(n+1)c_n}{2n+3}\right\},\ n=1\div\infty\,, \tag{45}$$

where

$$\beta_{1,n}^{+(j)}=-n\left(n+3-4\nu_j\right);\ \beta_{2,n}^{+(j)}=n+4\nu_j-4\,;\ \beta_{1,n}^{-(j)}=(n+1)\left(n+4\nu_j-2\right);\ \beta_{2,n}^{-(j)}=n-4\nu_j+5\,;$$

$$\rho_{1,n}^{+(j)}=n\left(n^2+3n-2\nu_j\right);\ \rho_{2,n}^{+(j)}=-\left(n^2+2\nu_j-2\right);\ \rho_{1,n}^{-(j)}=(n+1)\left(n^2-n-2\nu_j-2\right);\ \rho_{2,n}^{-(j)}=n^2+2n+2\nu_j-1\,;$$

$$u_{n,k}^{+(1)}=C_n^k\omega_{n+3,k+3}\,;\ u_{n,k}^{-(1)}=C_k^n\omega_{k-2,n-2}\,;\ u_{n,k}^{+(2)}=C_{n-1}^{k-1}\gamma_{k,n}^{+(2)}\omega_{n+1,k+1}\,;\ u_{n,k}^{-(2)}=C_{k+1}^{n+1}\gamma_{k,n}^{-(2)}\omega_{k,n}\,;\ u_{n,k}^{+(3)}=C_n^{k-1}\gamma_{k,n}^{+(1)}\omega_{n+3,k+1}\,;$$

$$u_{n,k}^{-(3)}=C_{k+1}^{n}\gamma_{k,n}^{-(1)}\omega_{k,n-2}\,;\ u_{n,k}^{+(4)}=C_n^k\gamma_{k,n}^{+(2)}\omega_{n+1,k+1}\,;\ u_{n,k}^{-(4)}=C_k^n\gamma_{k,n}^{-(2)}\omega_{k,n}\,;\ u_{n,k}^{+(5)}=C_{n+1}^k\lambda_{k,n}^{+(1)}\omega_{n+3,k+1}\,;$$

$$u_{n,k}^{-(5)}=C_{k+1}^n\lambda_{k,n}^{-(1)}\omega_{k,n-2}\,.$$

After eliminating unknowns $b_{i,n}$ and some transformations, the system (40) – (45) can be represented as

$$\xi_na_{1,n}^{(1)}+\eta_na_{2,n}^{(1)}+d_n^{-(1)}\sum_{k=1}^{n}u_{n,k}^{+(2)}a_{2,k}^{(2)}=f_n^{(1)}+(n+1)f_n^{(2)}-\tilde{\alpha}_1(n-1)t_n^{(1)}+2\tilde{\alpha}_1(n-1)\sum_{k=0}^{n}u_{n,k}^{+(4)}\frac{t_k^{(2)}}{2k-1},\ n=1\div\infty\,; \tag{46}$$

$$-d_n^{+(1)}a_{2,n}^{(1)}+(2n+1)(n+2)\sum_{k=0}^{n}u_{n,k}^{+(1)}a_{1,k}^{(2)}+n(n+2)(2n-1)\sum_{k=1}^{n}u_{n,k}^{+(2)}a_{2,k}^{(2)}+(2n+1)(n+2)\sum_{k=1}^{n+1}u_{n,k}^{+(3)}a_{2,k}^{(2)}=f_n^{(1)}-$$

$$-nf_n^{(2)}+\tilde{\alpha}_1\frac{2(n+2)t_n^{(1)}}{2n+3}+\tilde{\alpha}_1(n+2)(2n-1)\sum_{k=0}^{n}u_{n,k}^{+(4)}\frac{t_k^{(2)}}{2k-1}+\tilde{\alpha}_1(2n+1)(n+2)\sum_{k=1}^{n+1}u_{n,k}^{+(5)}\frac{t_k^{(2)}}{2k-1},\ n=0\div\infty\,; \tag{47}$$

$$a_{2,n}^{(2)}+\frac{(1-G_{21})\xi_n}{\Delta_n^{(5)}}\sum_{k=n}^{\infty}u_{n,k}^{-(1)}a_{1,k}^{(1)}+\frac{(1-G_{21})\eta_n}{\Delta_n^{(5)}}\sum_{k=n}^{\infty}u_{n,k}^{-(2)}a_{2,k}^{(1)}+\frac{(1-G_{21})\xi_n}{\Delta_n^{(5)}}\sum_{k=n-1}^{\infty}u_{n,k}^{-(3)}a_{2,k}^{(1)}=$$

$$=\tilde{\alpha}_1\frac{(1-G_{21})\xi_n}{(2n+1)\Delta_n^{(5)}}\frac{2t_n^{(2)}}{2n-1}-\tilde{\alpha}_1\frac{(1-G_{21})\eta_n}{(n+1)\Delta_n^{(5)}}\sum_{k=n}^{\infty}\frac{u_{n,k}^{-(4)}t_k^{(1)}}{2k+3}-\tilde{\alpha}_1\frac{(1-G_{21})\xi_n}{\Delta_n^{(5)}}\sum_{k=n-1}^{\infty}\frac{u_{n,k}^{-(5)}t_k^{(1)}}{2k+3},\ n=1\div\infty\,; \tag{48}$$

$$a_{1,n}^{(2)}+\frac{n(2n-1)}{2n+1}a_{2,n}^{(2)}-\frac{\Delta_n^{(4)}}{(2n+1)\Delta_n^{(3)}}\sum_{k=n}^{\infty}u_{n,k}^{-(2)}a_{2,k}^{(1)}=\tilde{\alpha}_1\frac{t_n^{(2)}}{2n+1}+\tilde{\alpha}_1\frac{2}{2n+1}\sum_{k=n}^{\infty}u_{n,k}^{-(4)}\frac{t_k^{(1)}}{2k+3}+$$

$$+\tilde{\alpha}_2G_{21}\frac{(2-2\nu_2)g_nR_2^2}{(2n+5)\Delta_n^{-(2)}\Delta_n^{(3)}}-\tilde{\alpha}_2G_{21}\frac{(4-4\nu_2)c_n}{\Delta_n^{-(2)}\Delta_n^{(3)}},\ n=0\div\infty\,; \tag{49}$$

$$b_{i,n}=\vartheta_n^{(i,1)}a_{1,n}^{(2)}+\vartheta_n^{(i,2)}a_{2,n}^{(2)}+\delta_{i,1}\sum_{k=n}^{\infty}u_{n,k}^{-(1)}a_{1,k}^{(1)}+\sum_{k=n}^{\infty}\psi_{n,k}^{(i)}a_{2,k}^{(1)}+\delta_{i,1}\sum_{k=n-1}^{\infty}u_{n,k}^{-(3)}a_{2,k}^{(1)}+\sigma_n^{(i)}\overline{t}_n^{(2)}+\sum_{k=n}^{\infty}\overline{\sigma}_{n,k}^{(i)}\overline{t}_k^{(1)}+$$

$$+\delta_{i,1}\sum_{k=n-1}^{\infty}\frac{u_{n,k}^{-(5)}}{2k+3}\overline{t}_k^{(1)}+\upsilon_n^{(i,1)}\overline{g}_n+\upsilon_n^{(i,2)}\overline{c}_n,\ n\in\mathbb{R}\colon n\geq\delta_{i,1}\,, \tag{50}$$

where

$$\Delta_n^{-(j)}=2\left[\left(3-4\nu_j\right)n+1-2\nu_j\right],\ j=1,2\,;\ d_n^{\pm(j)}=2\left[n^2+\left(1\pm2\nu_j\right)n+1\pm\nu_j\right],\ j=1,2\,;$$

$$\Delta_n^{(5)}=d_n^{-(1)}+G_{21}(n-1)\left(\Delta_n^{-(1)}+2\right);\ \Delta_n^{(3)}=n+2+G_{21}\frac{d_n^{+(2)}}{\Delta_n^{-(2)}},\ \Delta_n^{(4)}=d_n^{+(1)}-G_{21}d_n^{+(2)}\frac{\Delta_n^{-(1)}}{\Delta_n^{-(2)}}\,;\ G_{21}=\frac{G_2}{G_1}\,;$$

$$\xi_n=(n-1)(2n+1),\ \eta_n=(n-1)(n+1)(2n+3)\,;$$

$$\vartheta_n^{(1,1)} = \left[1 - \frac{(2n+1)}{\Delta_n^{-(2)}}\beta_{2,n}^{-(2)}\right],\ \vartheta_n^{(1,2)} = \left[\beta_{2,n}^{+(1)} - \frac{n(2n-1)}{\Delta_n^{-(2)}}\beta_{2,n}^{-(2)}\right],\ \psi_{n,k}^{(1)} = \left[\beta_{2,n}^{-(1)} - \frac{\Delta_n^{-(1)}}{\Delta_n^{-(2)}}\beta_{2,n}^{-(2)}\right]u_{n,k}^{-(2)};$$

$$\sigma_n^{(1)} = -\left[\frac{1}{2n-1} - \frac{\beta_{2,n}^{-(2)}}{\Delta_n^{-(2)}}\right],\ \bar{\sigma}_{n,k}^{(1)} = \left[1 + \frac{2\beta_{2,n}^{-(2)}}{\Delta_n^{-(2)}}\right]\frac{u_{n,k}^{-(4)}}{2k+3};\ \sigma_n^{(1)} = -\left[\frac{1}{2n-1} - \frac{\beta_{2,n}^{-(2)}}{\Delta_n^{-(2)}}\right],\ \bar{\sigma}_{n,k}^{(1)} = \left[1 + \frac{2\beta_{2,n}^{-(2)}}{\Delta_n^{-(2)}}\right]\frac{u_{n,k}^{-(4)}}{2k+3};$$

$$\upsilon_n^{(1,1)} = \frac{1}{4(2n+3)(2n+5)}\left[1 + 4\frac{\beta_{2,n}^{-(2)}}{\Delta_n^{-(2)}}\right],\ \upsilon_n^{(1,2)} = -\frac{1}{2n+3}\left[1 + 2\frac{\beta_{2,n}^{-(2)}}{\Delta_n^{-(2)}}\right];$$

$$\sigma_n^{(2)} = -\frac{1}{\Delta_n^{-(2)}},\ \bar{\sigma}_{n,k}^{(2)} = -\frac{2}{\Delta_n^{-(2)}}\frac{u_{n,k}^{-(4)}}{2k+3};\ \upsilon_n^{(2,1)} = -\frac{1}{(2n+3)(2n+5)\Delta_n^{-(2)}},\ \upsilon_n^{(2,2)} = \frac{2}{(2n+3)\Delta_n^{-(2)}}.$$

We will analyze the solvability of the system in the Hilbert space $l_2^4 = l_2 \times l_2 \times l_2 \times l_2$.

Theorem 3. *The operators of the system (46) – (49) is Fredholm operators in the space* l_2^4 *under the condition* $z_{12} + R_2 < R_1$.

Proof. Let us consider separately the equations of the system (46) – (49) at $n = 0$ and $n = 1$. When $n = 0$ there are only equations (47), (49) for determining $a_{2,0}^{(1)}$ and $a_{1,0}^{(2)}$. The coefficients $a_{1,0}^{(1)}$ and $a_{2,0}^{(2)}$ are arbitrary (chosen to be zero), since they are at zero solutions. When $n = 1$ from (46) and the static conditions $f_1^{(1)} + 2f_1^{(2)} = 0$ it follows $a_{2,1}^{(2)} = 0$, from (47) we find $a_{2,1}^{(1)}$, (48) is satisfied automatically, from (49) we find $a_{1,1}^{(2)}$, because the other unknown is $a_{2,1}^{(2)} = 0$, the coefficient $a_{1,1}^{(1)}$ is chosen arbitrarily, since the general solution depends on an arbitrary rigid displacement vector. Next, we will consider the equations of the system (46) – (49) when $n \geq 2$. Let us introduce the following notation:

$$\bar{a}_1^{(1)} = \left(a_{1,n}^{(1)}\right)_{n=2}^{\infty},\ \bar{a}_2^{(1)} = \left(a_{2,n}^{(1)}\right)_{n=2}^{\infty},\ \bar{a}_1^{(2)} = \left(a_{1,n}^{(2)}\right)_{n=2}^{\infty},\ \bar{a}_2^{(2)} = \left(a_{2,n}^{(2)}\right)_{n=2}^{\infty};\ a = \left(\bar{a}_1^{(1)}, \bar{a}_2^{(1)}, \bar{a}_1^{(2)}, \bar{a}_2^{(2)}\right)^T$$

Let us divide both parts of the system of equations (46) by a non-zero coefficient ξ_n, and divide the system (47) by $-d_n^{+(1)}$. After that, we write the resulting system in matrix form

$$\Phi a = z, \tag{51}$$

where

$$\Phi = S + U,\ S = \mathbf{diag}(S_1, S_2),\ S_1 = \begin{pmatrix} I & D_{12}^{(1)} \\ 0 & I \end{pmatrix},\ S_2 = \begin{pmatrix} 0 & I \\ I & D_{22}^{(2)} \end{pmatrix},$$

$$U = \text{adiag}(U_1, U_2),\ U_1 = \begin{pmatrix} 0 & U_{12}^{(2)} \\ U_{21}^{(2)} & U_{22}^{(2)} \end{pmatrix},\ U_2 = \begin{pmatrix} U_{11}^{(1)} & U_{12}^{(1)} \\ 0 & U_{22}^{(1)} \end{pmatrix},$$

$z = \left(\bar{z}_1^{(1)}, \bar{z}_2^{(1)}, \bar{z}_1^{(2)}, \bar{z}_2^{(2)}\right)^T$ – vector of the right-hand sides of the transformed system, $\text{diag}(\cdot)$ and $\text{adiag}(\cdot)$ are diagonal and anti-diagonal matrices respectively, I is the identity matrix, $D_{12}^{(1)}, D_{22}^{(2)}$ are some diagonal matrices. Here, the system (46) – (49) is written only structurally, since the matrix elements of the corresponding matrices follow from the system itself.

The operator S in the space l_2^4 has a bounded inverse operator $S^{-1} = \text{diag}\left(S_1^{-1}, S_2^{-1}\right)$, where

$$S_1^{-1} = \begin{pmatrix} I & -D_{12}^{(1)} \\ 0 & I \end{pmatrix},\ S_2^{-1} = \begin{pmatrix} -D_{22}^{(2)} & I \\ I & 0 \end{pmatrix}.$$

For the operator U, just as in Theorem 1, its compactness is proved. Then, by the *theorem of S. M. Nikolsky*, the operator Φ is a Fredholm operator.

Statement of an equivalent optimal control problem. After solving the direct problem, we will solve the inverse problem, that is, find the function (9) that satisfies conditions (7), (8). First, let's transform the functional (7). To do this, we substitute the stress vector on surface Γ_2 into functional (7)

$$F\vec{U}_2(r_2, \theta_2) = 2G_2\sum_{n=0}^{\infty}\left[b_{1,n}(n-1)n + b_{2,n}\rho_{1,n}^{-(2)} - \frac{\left(n^2 - n - 8\right)\bar{g}_n}{4(2n+3)(2n+5)} + \frac{\left(n^2 - n - 4\right)\bar{c}_n}{2n+3}\right]P_n(\cos\theta_2)\vec{e}_{r_2} +$$

$$+2G_2\sum_{n=0}^{\infty}\left[b_{1,n}\left(n-1\right)+b_{2,n}\rho_{2,n}^{-(2)}-\frac{\left(n+3\right)\overline{g}_n}{4\left(2n+3\right)\left(2n+5\right)}+\frac{\left(n+1\right)\overline{c}_n}{2n+3}\right]P_n^1\left(\cos\theta_2\right)\vec{e}_{\theta_2}\,, \tag{52}$$

where the coefficients g_n, c_n are replaced by normalized coefficients $\overline{g}_n=\tilde{\alpha}_2 g_n R_2^2$, $\overline{c}_n=\tilde{\alpha}_2 c_n$. The value of the objective functional (7) on the vector function (52) is a quadratic form on the infinite-dimensional space of numerical sequences $\left\{b_{1,n}, b_{2,n}, \overline{g}_n, \overline{c}_n\right\}_{n=0}^{\infty}$

$$J\left[\overline{g}\right]=\frac{1}{\left|\Gamma_2\right|}\int_{\Gamma_2}\left|F\vec{U}_2\right|^2 ds=2G_2^2\sum_{j=1}^{2}\sum_{n=0}^{\infty}\left[\sum_{i=1}^{2}\varepsilon_n^{(j,i)}b_{i,n}+\tau_n^{(j)}\overline{g}_n+\mu_n^{(j)}\overline{c}_n\right]^2, \tag{53}$$

where

$$\varepsilon_n^{(1,1)}=\frac{n\left(n-1\right)}{\sqrt{n+1/2}},\ \varepsilon_n^{(1,2)}=\frac{\rho_{1,n}^{-(2)}}{\sqrt{n+1/2}},\ \varepsilon_n^{(2,1)}=\left(n-1\right)\sqrt{\frac{2n\left(n+1\right)}{2n+1}},\ \varepsilon_n^{(2,2)}=\rho_{2,n}^{-(2)}\sqrt{\frac{2n\left(n+1\right)}{2n+1}},$$

$$\tau_n^{(1)}=-\frac{\left(n^2-n-8\right)}{4\left(2n+3\right)\left(2n+5\right)}\frac{1}{\sqrt{n+1/2}},\ \tau_n^{(2)}=-\frac{n+3}{4\left(2n+3\right)\left(2n+5\right)}\sqrt{\frac{2n\left(n+1\right)}{2n+1}},$$

$$\mu_n^{(1)}=\frac{n^2-n-4}{\left(2n+3\right)\sqrt{n+1/2}},\ \mu_n^{(2)}=\frac{n+1}{2n+3}\sqrt{\frac{2n\left(n+1\right)}{2n+1}},\ \overline{g}=\left(\overline{g}_k\right)_{k=0}^{\infty}.$$

The control constraint (8) can be written as follows:

$$\sum_{n=0}^{\infty}\frac{\overline{g}_n^2}{\left(2n+1\right)\left(2n+3\right)}=\frac{1}{3}\tilde{\alpha}_2^2 R_2^4 D^2. \tag{54}$$

Thus, the original optimal control problem is reduced to an equivalent problem in which the state of the object is determined by infinite systems of linear algebraic equations (18) – (20), (46) – (50), and optimal control must determine such coefficients $\left(\overline{g}_k\right)_{k=0}^{\infty}$ of the density of heat sources in the domain Ω_2 that give a minimum to the objective functional (53) and satisfy constraint (54).

Solution of the equivalent optimal control problem. The main problem of determining the minimum of the functional (53) under constraint (54) is that it is impossible to explicitly express the variables $\left\{b_{i,n}, \overline{c}_n\right\}_{n=0}^{\infty}$ of the quadratic functional through the optimization parameters $\left(\overline{g}_k\right)_{k=0}^{\infty}$. To solve this problem, a new method for parametric solution of infinite systems was proposed in [34]. We apply it to systems (18) – (20), (46) – (50).

To simplify further calculations, we will set $T_0=0$ (this does not affect the implementation of the method and the generality of the results). Due to the unique solvability of systems (18) – (19) and (46) – (49) with respect to $\left\{t_n^{(i)}\right\}_{n=0}^{\infty}$ and $\left\{a_{j,n}^{(i)}\right\}_{n=0}^{\infty}$, we can conclude that there exist linear infinite matrix operators $P^{(i)}=\left(p_{j,k}^{(i)}\right)_{j,k=0}^{\infty}$, $Q^{(m,i)}=\left(q_{j,k}^{(m,i)}\right)_{j,k=0}^{\infty}$, $R=\left(r_{j,k}\right)_{j,k=0}^{\infty}$, for which

$$b_{i,n}=\sum_{k=0}^{\infty}\left[p_{n,k}^{(i)}\overline{g}_k+\sum_{m=1}^{2}q_{n,k}^{(i,m)}f_k^{(m)}\right],\ \overline{c}_n=\sum_{k=0}^{\infty}r_{n,k}\overline{g}_k\,. \tag{55}$$

Note that the unknown coefficients of series (55) can be found using the formulas

$$p_{n,k}^{(i)}=\frac{\partial b_{i,n}}{\partial\overline{g}_k},\ q_{n,k}^{(i,m)}=\frac{\partial b_{i,n}}{\partial f_k^{(m)}},\ r_{n,k}=\frac{\partial\overline{c}_n}{\partial\overline{g}_k}. \tag{56}$$

Substitute formulas (55) into the functional (53)

$$J\left[\overline{g}\right]=2G_2^2\sum_{j=1}^{2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty}B_{n,k}^{(j)}\overline{g}_k+F_n^{(j)}\right]^2, \tag{57}$$

where

$$B_{n,k}^{(j)}=\sum_{i=1}^{2}\varepsilon_n^{(j,i)}p_{n,k}^{(i)}+\tau_n^{(j)}\delta_{n,k}+\mu_n^{(j)}r_{n,k}\,, \tag{58}$$

$$F_n^{(j)}=\sum_{i=1}^{2}\varepsilon_n^{(j,i)}\sum_{m=1}^{2}\sum_{k=0}^{\infty}q_{n,k}^{(i,m)}f_k^{(m)}\ . \quad (59)$$

After reducing the functional to a physically dimensionless form, we have:

$$\frac{1}{2}\sum_{j=1}^{2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty}B_{n,k}^{(j)}\overline{g}_k+F_n^{(j)}\right]^2\to\min\ , \quad (60)$$

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{\overline{g}_n^2}{(2n+1)(2n+3)}=\frac{1}{6}\tilde{\alpha}_2^2R_2^4D^2\ . \quad (61)$$

We will solve the problem (60), (61) for the conditional extremum by the Lagrange method, reducing it to the problem for the unconditional minimum of the functional

$$L[\overline{g}]=\frac{1}{2}\sum_{j=1}^{2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty}B_{n,k}^{(j)}\overline{g}_k+F_n^{(j)}\right]^2+\frac{\tilde{\lambda}}{2}\sum_{n=0}^{\infty}\frac{\overline{g}_n^2}{(2n+1)(2n+3)}\ . \quad (62)$$

Here $\bar{\lambda}$ is the Lagrange multiplier. The existence and uniqueness of the solution to problem 60), (61) will be proved further in Theorem 5. The necessary minimum condition of the functional (62) leads to the following system:

$$\left(\tilde{\lambda}\tilde{D}+B\right)\overline{g}=-F\ , \quad (63)$$

where

$$B=\left(\sum_{j=1}^{2}\sum_{n=0}^{\infty}B_{n,s}^{(j)}B_{n,k}^{(j)}\right)_{s,k=0}^{\infty}\ , \quad (64)$$

$$\tilde{D}=\mathbf{diag}\left((2s+1)^{-1}(2s+3)^{-1}\right)_{s=0}^{\infty},\quad F=\left(\sum_{j=1}^{2}\sum_{n=0}^{\infty}B_{n,s}^{(j)}F_n^{(j)}\right)_{s=0}^{\infty}\ . \quad (65)$$

The Lagrange multiplier is found from the additional condition (61).

Before examining the operator of system (63), we present a practical way of calculating the elements of the matrices B and F. Based on formulas (16) – (18), (46) – (51), we have $\left(s=0\div\infty,\ i=1,2\right)$

$$\frac{\partial\overline{t}_n^{(1)}}{\partial\overline{g}_s}+\sum_{k=0}^{n}u_{n,k}^{+}\frac{\partial\overline{t}_k^{(2)}}{\partial\overline{g}_s}=0\ ,\quad n=0\div\infty\ ; \quad (66)$$

$$\frac{\partial\overline{t}_n^{(2)}}{\partial\overline{g}_s}+K_n^{(1)}\sum_{k=n}^{\infty}u_{n,k}^{-}\frac{\partial\overline{t}_k^{(1)}}{\partial\overline{g}_s}=\frac{\tilde{\alpha}_1}{\tilde{\alpha}_2}K_n^{(2)}\frac{\delta_{n,s}}{2n+3}\ ,\quad n=0\div\infty\ ; \quad (67)$$

$$\frac{\partial\overline{c}_n}{\partial\overline{g}_s}=\frac{\tilde{\alpha}_2}{\tilde{\alpha}_1}\left(\frac{\partial\overline{t}_n^{(2)}}{\partial\overline{g}_s}+\sum_{k=n}^{\infty}u_{n,k}^{-}\frac{\partial\overline{t}_k^{(1)}}{\partial\overline{g}_s}\right)+\frac{\delta_{n,s}}{2(2n+3)}\ ,\quad n=0\div\infty\ , \quad (69)$$

$$\Phi\frac{\partial a}{\partial\overline{g}_s}=\frac{\partial z}{\partial\overline{g}_s}\ ,\quad \Phi\frac{\partial a}{\partial f_s^{(m)}}=\frac{\partial f_t}{\partial f_s^{(m)}}\ , \quad (70)$$

$$\frac{\partial b_{i,n}}{\partial\overline{g}_s}=\vartheta_n^{(i,1)}\frac{\partial a_{1,n}^{(2)}}{\partial\overline{g}_s}+\vartheta_n^{(i,2)}\frac{\partial a_{2,n}^{(2)}}{\partial\overline{g}_s}+\delta_{i,1}\sum_{k=n}^{\infty}u_{n,k}^{-(1)}\frac{\partial a_{1,k}^{(1)}}{\partial\overline{g}_s}+\sum_{k=n}^{\infty}\psi_{n,k}^{(i)}\frac{\partial a_{2,k}^{(1)}}{\partial\overline{g}_s}+\delta_{i,1}\sum_{k=n-1}^{\infty}u_{n,k}^{-(3)}\frac{\partial a_{2,k}^{(1)}}{\partial\overline{g}_s}+\sigma_n^{(i)}\frac{\partial\overline{t}_n^{(2)}}{\partial\overline{g}_s}+\sum_{k=n}^{\infty}\overline{\sigma}_{n,k}^{(i)}\frac{\partial\overline{t}_k^{(1)}}{\partial\overline{g}_s}+$$

$$+\delta_{i,1}\sum_{k=n-1}^{\infty}\frac{u_{n,k}^{-(5)}}{2k+3}\frac{\partial\overline{t}_k^{(1)}}{\partial\overline{g}_s}+\upsilon_n^{(i,1)}\delta_{n,s}+\upsilon_n^{(i,2)}\frac{\partial\overline{c}_n}{\partial\overline{g}_s}\ ,\quad n\in\mathbb{Z}\colon n\geq\delta_{i,1}\ , \quad (71)$$

$$\frac{\partial b_{i,n}}{\partial f_s^{(m)}}=\vartheta_n^{(i,1)}\frac{\partial a_{1,n}^{(2)}}{\partial f_s^{(m)}}+\vartheta_n^{(i,2)}\frac{\partial a_{2,n}^{(2)}}{\partial f_s^{(m)}}+\delta_{i,1}\sum_{k=n}^{\infty}u_{n,k}^{-(1)}\frac{\partial a_{1,k}^{(1)}}{\partial f_s^{(m)}}+\sum_{k=n}^{\infty}\psi_{n,k}^{(i)}\frac{\partial a_{2,k}^{(1)}}{\partial f_s^{(m)}}+\delta_{i,1}\sum_{k=n-1}^{\infty}u_{n,k}^{-(3)}\frac{\partial a_{2,k}^{(1)}}{\partial f_s^{(m)}}\ ,\ n\in\mathbb{Z}\colon n\geq\delta_{i,1}\ . \quad (72)$$

Here the system (16) – (18) is written for physically dimensionless coefficients $\overline{t}_n^{(i)}=\tilde{\alpha}_1 t_n^{(i)}$, $\delta_{i,k}$ – Kronecker's delta symbol.

Remark. In order to find the elements of matrices (64), (65), it is necessary to solve the infinite systems (66), (67), (70). The matrices of these systems are the same as those of systems (16), (17), (46) – (49), only their right-hand sides change.

For further, we will enter the notation

$$\bar{g}_n = \varsigma_n \tilde{g}_n \,, \; \tilde{B} = \left(\sum_{j=1}^{2} \sum_{n=0}^{\infty} \tilde{B}_{n,m}^{(j)} \tilde{B}_{n,k}^{(j)} \right)_{m,k=0}^{\infty} \,, \; \tilde{B}_{n,m}^{(j)} = B_{n,m}^{(j)} \varsigma_m \,, \; \tilde{F} = \left(\sum_{j=1}^{2} \sum_{n=0}^{\infty} \tilde{B}_{n,m}^{(j)} F_n^{(j)} \right)_{m=0}^{\infty} \,, \; \varsigma_n = \sqrt{(2n+1)(2n+3)} \,.$$

In the new notation, system (63) can be written in the form

$$\left(\tilde{\lambda} I + \tilde{B} \right) \tilde{g} = -\tilde{F} \,. \tag{73}$$

with a constraint

$$\sum_{n=0}^{\infty} \tilde{g}_n^2 = \frac{1}{3} \tilde{\alpha}_2^2 R_2^4 D^2 \,. \tag{74}$$

Theorem 4. *When the conditions of Theorem 3 are met, the matrix* $\tilde{B}$ *of system (73) defines a symmetric, positive definite, compact operator in space* l_2 .

Proof. The symmetry of the matrix $\tilde{B}$ is obvious. Besides, it is positive since for any vector $x = (x_k)_{k=0}^{\infty} \in l_2$ quadratic form

$$\left(\tilde{B} x, x \right) = \sum_{m,k=0}^{\infty} x_m x_k \sum_{j=1}^{2} \sum_{n=0}^{\infty} \tilde{B}_{n,m}^{(j)} \tilde{B}_{n,k}^{(j)} = \sum_{j=1}^{2} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \tilde{B}_{n,k}^{(j)} x_k \right)^2 \geq 0 \,.$$

Let us show that matrix $\tilde{B}$ is positive definite. Let's put $\left(\tilde{B} \tilde{g}, \tilde{g} \right) = 0$. Then

$$\sum_{j=1}^{2} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \tilde{B}_{n,k}^{(j)} \tilde{g}_k \right)^2 = \sum_{j=1}^{2} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} B_{n,k}^{(j)} \bar{g}_k \right)^2 = 0 \,.$$

On the other hand, the above quadratic form in the absence of external load in accordance with formula (57) coincides with the target functional up to a factor, therefore,

$$J[\bar{g}] = \frac{1}{|\Gamma_2|} \int_{\Gamma_2} \left| F \vec{U}_2 \right|^2 ds = 0 \,,$$

whence

$$\left(F \vec{U}_2 \right)_{|\Gamma_2} = 0 \,.$$

The last condition is possible only in the absence of heat sources in the domain Ω_2, i.e. when $\bar{g} = (\bar{g}_k)_{k=0}^{\infty} = 0$ or $\tilde{g} = 0$. Thus, matrix $\tilde{B}$ is positive definite.

Let us prove that the matrix $\tilde{B}$ defines a compact operator in the Hilbert space l_2 . First, we will prove the compactness of the operator defined by a matrix with elements $\tilde{B}_{n,m}^{(j)}$. Since it is impossible to strictly estimate the matrix elements of this matrix, we will apply the method proposed in the article [34]. In this work it was further developed. Let us introduce the notation for the following infinite matrices:

$$T_g^{(i)} = \left(\frac{\partial \bar{t}_n^{(i)}}{\partial \bar{g}_m} \right)_{n,m=0}^{\infty} \,, \; C_g = \left(\frac{\partial \bar{c}_n}{\partial \bar{g}_m} \right)_{n,m=0}^{\infty} \,, \; A_i^{(j)} = \left(\frac{\partial a_{i,n}^{(j)}}{\partial \bar{g}_m} \right)_{n,m=0}^{\infty} \,, \; B_g^{(i)} = \left(\frac{\partial b_{i,n}}{\partial \bar{g}_m} \right)_{n,m=0}^{\infty} \,.$$

Systems (66) – (72) can be written in matrix form

$$T_g^{(1)} = -U^+ T_g^{(2)} \,, \; T_g^{(2)} + U_1^- T_g^{(1)} = D_1 \,, \; C_g = \frac{\tilde{\alpha}_2}{\tilde{\alpha}_1} T_g^{(2)} + U_2^- T_g^{(1)} + D_2 \,, \tag{75}$$

$$A_1^{(1)} + D_{12}^{(1)} A_2^{(1)} + U_{12}^{(2)} A_2^{(2)} = D_g^{(1,1)} T_g^{(1)} + U_{11}^+ T_g^{(2)} \,, \tag{76}$$

$$A_2^{(1)} + U_{21}^{(2)} A_1^{(2)} + U_{22}^{(2)} A_2^{(2)} = D_g^{(2,1)} T_g^{(1)} + U_{12}^+ T_g^{(2)} \,, \tag{77}$$

$$A_2^{(2)} + U_{11}^{(1)} A_1^{(1)} + U_{12}^{(1)} A_2^{(1)} = D_g^{(1,2)} T_g^{(2)} + U_{21}^- T_g^{(1)} \,, \tag{78}$$

$$A_1^{(2)} + D_{22}^{(2)} A_2^{(2)} + U_{12}^{(1)} A_2^{(1)} = D_g^{(2,2)} T_g^{(2)} + U_{22}^- T_g^{(1)} + D_g^{(1)} + D_g^{(2)} C_g \,, \tag{79}$$

$$B_g^{(i)} = D_b^{(i,1)} A_1^{(2)} + D_b^{(i,2)} A_2^{(2)} + U_b^{(i,1)} A_1^{(1)} + U_b^{(i,2)} A_2^{(1)} + D_\sigma^{(i)} T_g^{(2)} + U_\sigma^{(i)} T_g^{(1)} + D_b^{(1)} + D_b^{(2)} C_g \,. \tag{80}$$

For reasons of compactness, the elements of the coefficient matrices are not presented here. They can be reconstructed from systems (66) – (72). They are not needed in this context. Above, the letter D with different indices denotes diagonal operators, and the letter U with indices denotes operators with matrix elements $\left(u_{n,k}\right)_{n,k=0}^{\infty}$ for which the series

$$\sum_{n,k=0}^{\infty} n^s k^r \left|u_{n,k}\right| < \infty$$

converges, where s and r are any non-negative numbers. The last condition means that all operators U are not only bounded, but also compact. Matrices $T_g^{(i)}$, C_g, $A_i^{(j)}$ define bounded operators acting in the space l_2, which follows from the correct solvability of systems (66) – (69). From equation (58) follows the matrix equality

$$\left(\tilde{B}_{n,k}^{(j)}\right)_{n,k=0}^{\infty} = \operatorname{diag}\left(\varsigma_n \varepsilon_n^{(j,1)}\right) B_g^{(1)} + \operatorname{diag}\left(\varsigma_n \varepsilon_n^{(j,2)}\right) B_g^{(2)} + \operatorname{diag}\left(\varsigma_n \tau_n^{(j)}\right) + \operatorname{diag}\left(\varsigma_n \mu_n^{(j)}\right) C_g , \qquad (81)$$

If we substitute all the matrices obtained by inverting systems (66) – (69) into formula (81), we obtain two groups of terms. The first includes products of bounded and compact operators, i.e., compact operators. The second includes diagonal operators. Then, to establish the compactness of operators $\left(\tilde{B}_{n,k}^{(j)}\right)_{n,k=0}^{\infty}$, it suffices to prove the compactness of their diagonal operators. Omitting the intermediate, rather cumbersome transformations, we present the matrix elements of the diagonal matrices that make up the matrices $\left(\tilde{B}_{n,k}^{(1)}\right)_{n,k=0}^{\infty}$ and $\left(\tilde{B}_{n,k}^{(2)}\right)_{n,k=0}^{\infty}$, respectively

$$D\tilde{B}_{n,m}^{(1)} = -\left\{ \frac{\tilde{\alpha}_1}{\tilde{\alpha}_2} \frac{\left(4-4\nu_1\right) n\left(n-1\right) K_n^{(2)}}{\left(2n+3\right)\Delta_n^{(5)}} + \frac{\left(4-4\nu_2\right)\left(n+1\right)\left(n+2\right)\left[\left(2n+5\right) K_n^{(2)}+1\right]}{\left(2n+3\right)\left(2n+5\right)\Delta_n^{-(2)}\Delta_n^{(3)}} \right\} \frac{\varsigma_n \delta_{n,m}}{\sqrt{n+1/2}} , \qquad (81)$$

$$D\tilde{B}_{n,m}^{(2)} = -\left\{ \frac{\tilde{\alpha}_1}{\tilde{\alpha}_2} \frac{\left(4-4\nu_1\right)\left(n-1\right) K_n^{(2)}}{\left(2n+3\right)\Delta_n^{(5)}} - \frac{\left(4-4\nu_2\right)\left(n+2\right)\left[\left(2n+5\right) K_n^{(2)}+1\right]}{\left(2n+3\right)\left(2n+5\right)\Delta_n^{-(2)}\Delta_n^{(3)}} \right\} \sqrt{\frac{2n\left(n+1\right)}{2n+1}} \varsigma_n \delta_{n,m} . \qquad (82)$$

As can be seen from formulas (81), (82), the matrix elements of the diagonal matrices $\left(D\tilde{B}_{n,k}^{(j)}\right)_{n,k=0}^{\infty}$ have order $O\left(n^{-3/2}\right)$ for $n \to \infty$, which guarantees the compactness of the operators defined by these matrices.

Consequently, the theorem is completely proven.

Remark. Using the same ideas, it is possible to prove that if $f^{(j)} \in l_2$ the column $\tilde{F} \in l_2$.

The further solution of the problem is based on the spectral method. It follows from the properties of the operator $\tilde{B}$ that its spectrum consists of a counted sequence of positive eigenvalues

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge \cdots ,$$

that converges to zero. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a complete orthonormal system of eigenvectors of the operator $\tilde{B}$ in the space l_2 corresponding to the eigenvalues of $\{\lambda_n\}_{n=1}^{\infty}$. Let us denote by $F_n = \left(\tilde{F}, \varphi_n\right)$ the Fourier coefficients of the expansion of the vector $\tilde{F}$ into a Fourier series in a system of functions $\{\varphi_n\}_{n=1}^{\infty}$.

Theorem 5. *Let the conditions of Theorem 3 hold. If*

$$\sum_{n=1}^{\infty} \frac{F_n^2}{\lambda_n^2} \le \frac{1}{3} \tilde{\alpha}_2^2 R_2^4 D^2 , \qquad (83)$$

then there is a solution to equation (73) at $\tilde{\lambda} = 0$

$$\tilde{g} = -\sum_{n=1}^{\infty} \frac{F_n}{\lambda_n} \varphi_n \in l_2 , \qquad (84)$$

and constraint (74) is not taken into account. If

$$\sum_{n=1}^{\infty} \frac{F_n^2}{\lambda_n^2} > \frac{1}{3} \tilde{\alpha}_2^2 R_2^4 D^2 , \qquad (85)$$

then under $\tilde{\lambda} > 0$ *there is a unique solution of problem (73), (74) in space* l_2 *of the form*

$$\tilde{g} = -\sum_{n=1}^{\infty} \frac{F_n}{\tilde{\lambda} + \lambda_n} \varphi_n . \qquad (86)$$

The proof of this theorem is completely analogous to the proof of the corresponding theorem in paper [34], and is therefore not given here.

Remark. When condition (83) is fulfilled, optimal control (84) is a solution of the original problem (1) – (7) without restriction (8).

Computer experiment. The following materials were used in the numerical solution of the problem: steel for the ball Ω_1 and brass for the inclusion Ω_2. These materials have the following thermomechanical characteristics: $\left(G_1 = 82.0\,\text{GPa},\ \nu_1 = 0.28,\ \alpha_1 = 13.010^{-6}\ {}^\circ C^{-1},\ k_1 = 45.4\,\text{W}/\left(\text{M}\cdot{}^\circ C\right)\right)$ for the ball Ω_1 and $\left(G_2 = 35.2\,\text{GPa},\ \nu_1 = 0.35,\ \alpha_2 = 18.7\cdot 10^{-6}\ {}^\circ C^{-1},\ k_2 = 85.5\,\text{W}/\left(\text{M}\cdot{}^\circ C\right)\right)$. The value of the constant $k_2 D$ was chosen equal to $4.28\cdot 10^4\ \text{W/M}^3$. According to the algorithm presented above, when solving the optimal control problem, it is necessary to form and solve a series of infinite systems of linear algebraic equations: (66) – (69), (70), (73). To determine the optimal temperature and deformation fields, using the found optimal value of the vector x, systems (18), (19), and (46) – (49) are solved. Since the Fredholm's property of the operators has been proven for all of the systems listed above, the systems can be solved numerically using the reduction method. At the same time, the numerical algorithm is correct, i.e., with an unlimited increase in the reduction parameter, the numerical solution converges to the exact one.

To test the universality of the proposed approach, three different types of external load were used in a computer experiment.

The first type of external load. In the first case, we will consider a uniform distribution of the load on the surface Γ_1 of the ball

$$\vec{f}\left(\theta_1\right) = 2G_1\sigma\vec{e}_{r_1}\ ,$$

where the parameter σ was selected so that the external load does not exceed the elastic limit. Figures 1 and 2 show the optimal distributions of the temperature fields in the sphere and in the inclusion and the reduced power of the sources in the inclusion, respectively, for the geometric parameters $R_2 / R_1 = 0.5$, $z_{12} / R_1 = 0.3$.

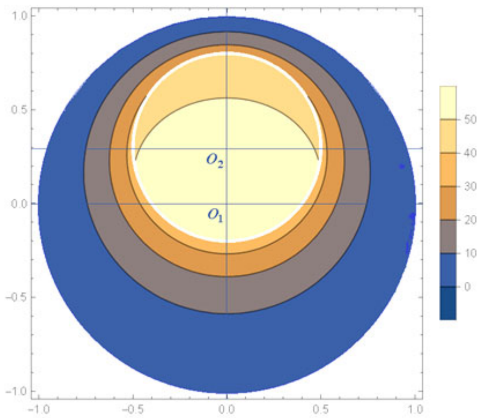

Fig. 1 – Optimal temperature distribution in the ball. The first type of loading.

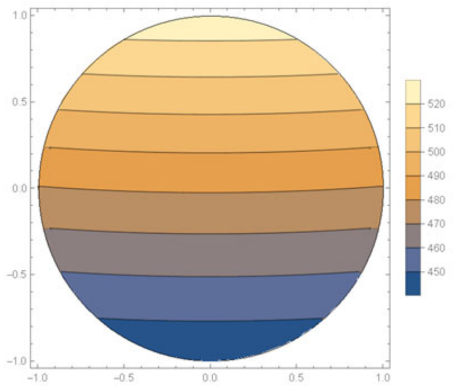

Fig. 2 – Optimal distribution of reduced power of sources in the inclusion. The first type of loading.

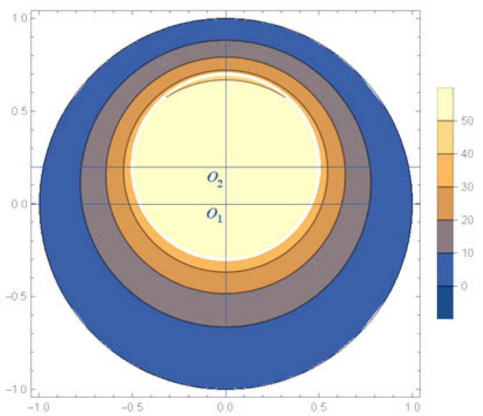

Fig. 3 – Optimal temperature distribution in the ball. The first type of loading.

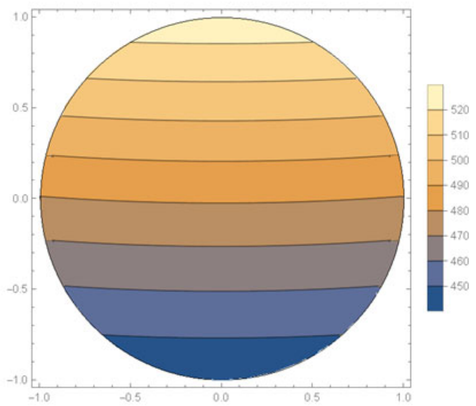

Fig. 4 – Optimal distribution of reduced power of sources in the inclusion. The first type of loading.

Figures 3, 4 show a similar distribution of temperature field and the reduced power of heat sources for geometric parameters $R_2 / R_1 = 0.5$, $z_{12} / R_1 = 0.2$.

Figures 5, 6 show a similar distribution of temperature field and the reduced power of heat sources for geometric parameters $R_2 / R_1 = 0.5$, $z_{12} / R_1 = 0.1$.

Fig. 7 shows graphs of the distribution of the optimal temperature on the surface of the inclusion for a ratio of the radii of the inclusion and the sphere equal to $R_2 / R_1 = 0.5$, and different displacements of the center of the inclusion relative to the center of the sphere. Since in the original formulation of the optimization problem the objective functional was a functional expressing the root-mean-square value of stresses on the inclusion surface, it is of interest to trace the distribution of normal (largest in modulus) stresses on this surface. Fig. 8 shows the distribution of normal stresses on the inclusion surface for the geometric parameters indicated in Fig. 7. The graphs presented in Fig. 8 show that optimal voltages when controlling the power of heat sources can reach maximum values that are almost an order of magnitude lower than the voltages in the absence of heat sources.

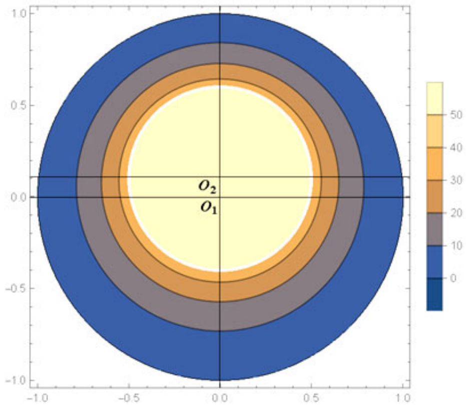

Fig. 5 – Optimal temperature distribution in the ball. The first type of loading.

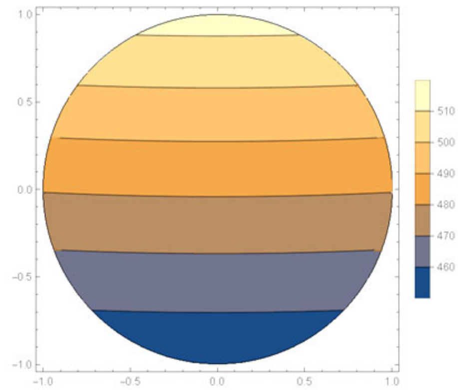

Fig. 6 – Optimal distribution of reduced power of sources in the inclusion. The first type of loading.

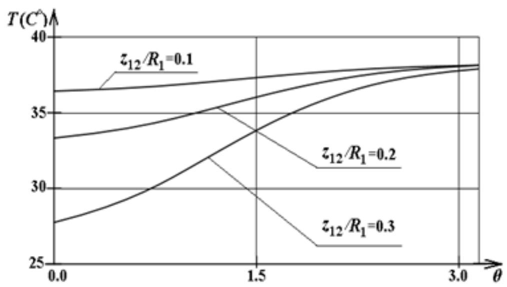

Fig. 7 – Optimal temperature distribution on the inclusion surface. The first type of loading.

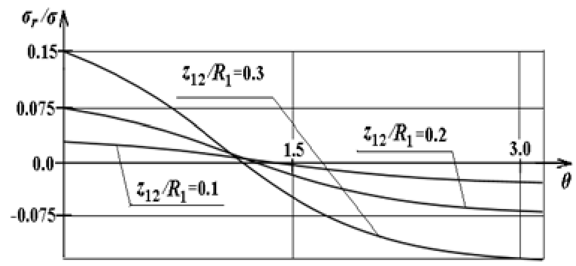

Fig. 8 – Optimal distribution of normal stresses on the inclusion surface. The first type of loading.

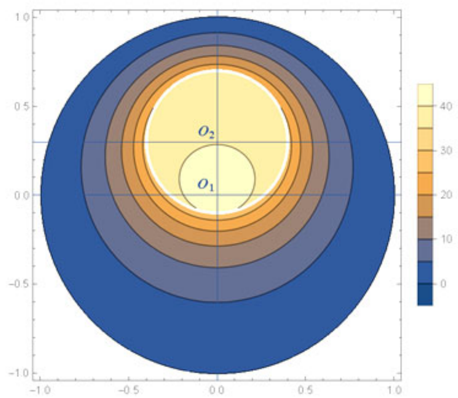

Fig. 9 – Optimal temperature distribution in the ball. The first type of loading.

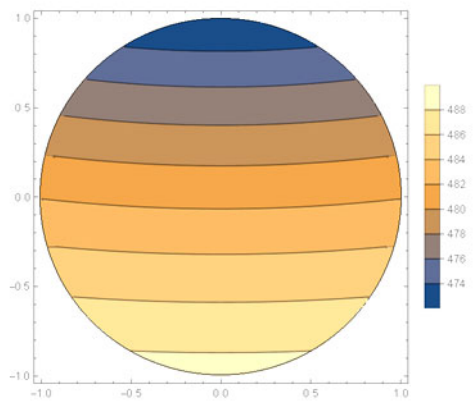

Fig. 10 – Optimal distribution of reduced power of sources in the inclusion. The first type of loading.

Figures 9, 10 and 11, 12 show the distributions of temperature fields and reduced powers of heat sources at a fixed distance $z_{12} / R_1 = 0.3$ between the centers O_1 and O_2 for different ratios R_2 / R_1. It is interesting to note that with such a change in the geometric parameters compared to those indicated above, the inclusion areas with maximum and minimum reduced power are reflected relative to the horizontal plane passing through the inclusion center.

The second type of external load. In the second case, the load on the surface Γ_1 was considered, which is given by the formula

$$\vec{f}(\theta_1) = 2G_1\sigma \sin\theta_1 \vec{e}_{r_1} .$$

Figures 13 and 14 show the optimal distributions of the temperature field in the sphere and in the inclusion, as well as the reduced power of the sources in the inclusion, respectively, for the geometric parameters $R_2 / R_1 = 0.5$, $z_{12} / R_1 = 0.3$ and the second type of loading.

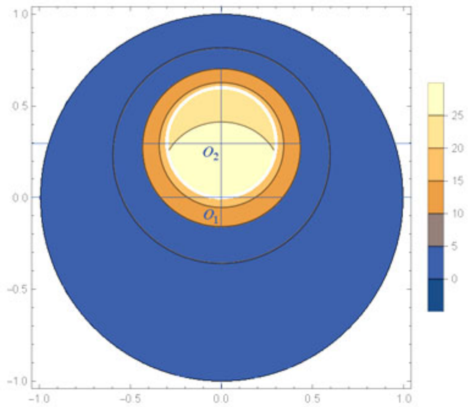

Fig. 11 – Optimal temperature distribution in the ball. The first type of loading.

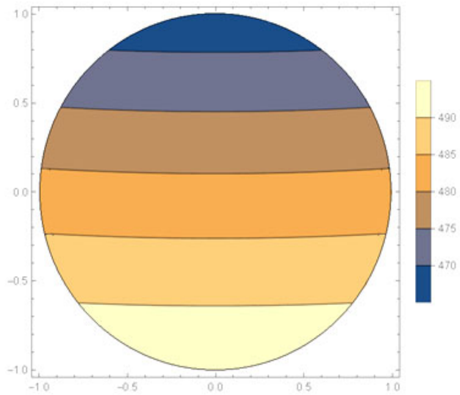

Fig. 12 – Optimal distribution of reduced power of sources in the inclusion. The first type of loading.

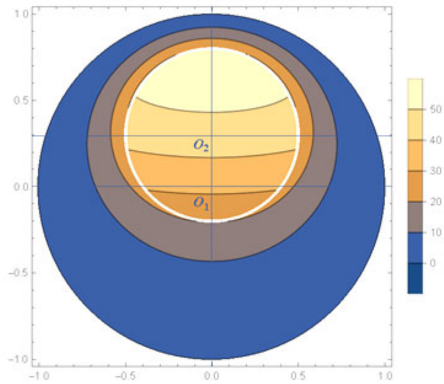

Fig. 13 – Optimal temperature distribution in the ball. The second type of loading.

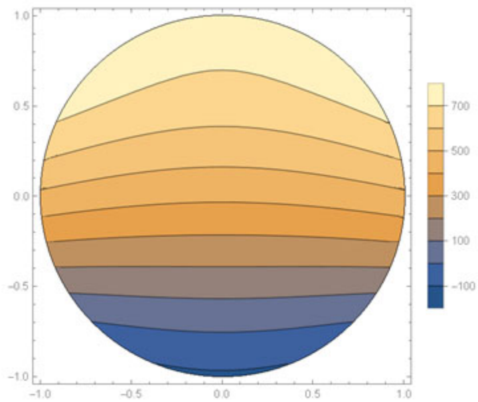

Fig. 14 – Optimal distribution of reduced power of sources in the inclusion. The second type of loading.

The third type of external load. In the third case, the load on the surface Γ_1 was considered, which is given by the formula

$$\vec{f}(\theta_1) = 2G_1\sigma\left[3\sin\theta_1 \cos\theta_1 \vec{e}_{r_1} + \sin^2\theta_1 \vec{e}_{\theta_1}\right].$$

Figures 15 and 16 show the optimal distributions of the temperature field in the sphere and in the inclusion, as well as the reduced power of the sources in the inclusion, respectively, for the geometric parameters $R_2 / R_1 = 0.5$, $z_{12} / R_1 = 0.3$ and the third type of loading.

Figures 17 and 18 show graphs of the optimal temperature and normal stresses on the inclusion surface for three types of external load at $R_2 / R_1 = 0.5$, $z_{12} / R_1 = 0.3$. Each type is indicated by the corresponding number on the graph.

Remark. The first two types represent a balanced load for the first component. The third type satisfies the balance condition for two components.

Note that for all types of loads, a continuous dependence of temperatures and stresses on geometric parameters is observed, which indirectly confirms the correctness of the solved inverse problems.

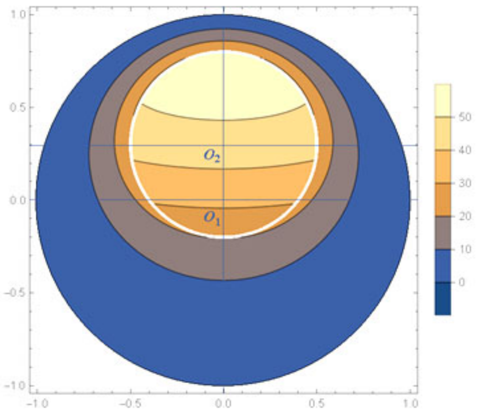

Fig. 15 – Optimal temperature distribution in the ball. The third type of loading.

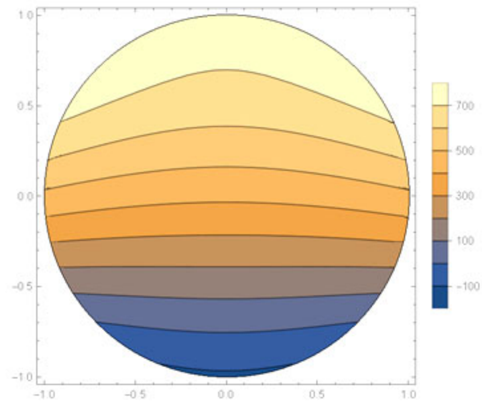

Fig. 16 – Optimal distribution of reduced power of sources in the inclusion. The third type of loading.

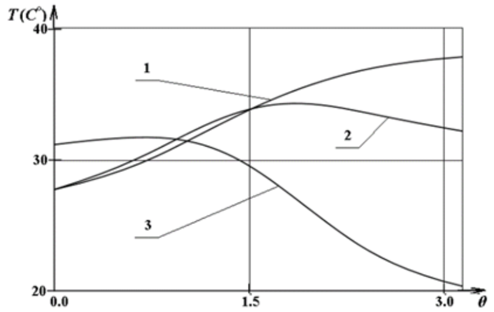

Fig. 17 – Graphs of the distribution of the optimal temperature on the inclusion surface for different types of external load.

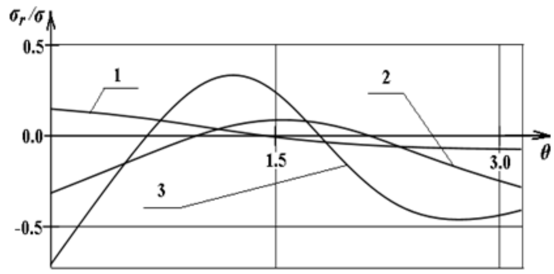

Fig. 18 – Graphs of the distribution of optimal normal stresses on the inclusion surface for different types of external load.

Additionally, the practical convergence of the reduction method was tested in a number of problems. Calculations have shown that the values of temperature and normal stress on the surface of the inclusion retain five upper significant digits after the point at a relative proximity of the surfaces of the sphere and the inclusion equal to

$$1 - z_{12} / R_1 - R_2 / R_1 = 0.2 ,$$

for the reduction parameter $n_{\max} = 20$ ($n_{\max} + 1$ is the order of each matrix block of the matrix Φ in formula (51)).

Prospects for further research. The method proposed in the work allows for further development both for bodies of other geometries and for problems of optimal control of other types of physical fields. Another direction of its development is related to non-axisymmetric problems.

Conclusions. As a result of the conducted research, the following conclusions can be drawn. The work presents a further development of the method for solving problems of optimal control of the thermoelastic state of a composite body using a temperature field, proposed in [34]. Unlike the work mentioned above, in which it was developed to control the surface temperature of the body, in this article the control is the power of distributed internal heat sources. As far as the authors know, this type of control is being considered for the first time. The method is demonstrated using the example of solving the problem of optimal control of the thermoelastic state of a loaded ball with an eccentric inclusion made of another material, in the region of which thermal field sources are distributed. The objective functional chosen for the problem is a functional that expresses the root-mean-square value of the stress at the inclusion surface. An additional constraint is imposed on the root-mean-square power of the heat sources. The solution to the problem is divided into two stages. At the first stage, the direct problem of modeling the thermoelastic state of a ball with inclusion is solved at a given temperature and external load at the boundary of the ball, and in the inclusion – the power of distributed heat sources. In this case, the generalized Fourier method is used, a modification of the apparatus of which was specially developed by the authors for the specified geometry of the body. As a result of the implementation of the first stage, the problem of modeling the thermoelastic state of the sphere is reduced to solving two infinite systems of linear algebraic equations to determine the parameters of the model, and the original optimal control problem is replaced by an equivalent problem in which the state of the object is already determined by the specified systems. In this case, the opti-

mization problem is posed with respect to a quadratic functional defined on the Cartesian product of Hilbert spaces of square-summable numerical sequences. The arguments of the functional are the solutions of the specified systems. The fundamental difficulty in solving the equivalent (inverse) problem is the impossibility of analytically expressing the solutions of the systems through optimization parameters. This paper further develops the method for parametrically solving infinite systems of linear algebraic equations proposed in [34]. As a result of its application, the quadratic functional was expressed in terms of optimization parameters. The problem of finding the functional's conditional extremum was solved using the Lagrange method. Its application led to an infinite system of linear algebraic equations with a quadratic constraint, which was solved using the spectral method. All intermediate stages of the implementation of the proposed method are strictly justified by five theorems proven in the work. The numerical implementation of the method was conducted as part of an extensive computer experiment, the results of which are presented in the paper. The paper presents images of the optimal temperature fields in the sphere and the powers of the distributed heat sources in the inclusion, as well as graphs of the optimal temperatures and normal stresses on the inclusion surface depending on the geometric parameters and three different types of external load. The analytical justification and calculations performed prove the correctness and effectiveness of the proposed method.

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Відомості про авторів / Information about authors

Ніколаєв Олексій Георгійович – доктор фізико-математичних наук, професор, професор кафедри вищої математики та системного аналізу, Національний аерокосмічний університет «Харківський авіаційний інститут», м. Харків; тел.: (067) 996-04-92; ORCID: https://orcid.org/0000-0002-7351-1392; e-mail: a.nikolaev@khai.edu.

Nikolaev Oleksii Georhiiovych – Doctor of Physical and Mathematical Sciences, Professor, Professor of Department of Higher Mathematics and System Analysis, National Aerospace University "Kharkiv Aviation Institute", Kharkiv; tel.: (067) 996-04-92; ORCID: https://orcid.org/0000-0002-7351-1392; e-mail: a.nikolaev@khai.edu.

Скіцка Марія Вікторівна – доктор філософії, доцент кафедри вищої математики та системного аналізу, Національний аерокосмічний університет «Харківський авіаційний інститут», м. Харків; тел.: (095) 596-30-79; ORCID: https://orcid.org/0009-0008-8945-6069; e-mail: m.skitska@khai.edu.

Skitska Mariia Viktorivna – PhD, Associate Professor of the Department of Higher Mathematics and System Analysis, National Aerospace University "Kharkiv Aviation Institute", Kharkiv; tel.: (095) 596-30-79; ORCID: https://orcid.org/0009-0008-8945-6069; e-mail: m.skitska@khai.edu.

Драшпуль Наталя Володимирівна – старший викладач кафедри вищої математики та системного аналізу, Національний аерокосмічний університет «Харківський авіаційний інститут», м. Харків; тел.: (095) 580-94-01; ORCID: https://orcid.org/0009-0003-8885-8262; e-mail: n.drashpul@khai.edu.

Drashpul Natalia Volodymyrivna – Senior Lecturer of the Department of Higher Mathematics and System Analysis, National Aerospace University "Kharkiv Aviation Institute", Kharkiv; tel.: (095) 580-94-01; ORCID: https://orcid.org/0009-0003-8885-8262; e-mail: n.drashpul@khai.edu.