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CLASSICAL PROBLEM ABOUT AN ELASTIC SPHERE WITH A SPHERICAL INCLUSION

For the first time, an exact analytically justified solution of the second axisymmetric boundary value problem of the theory of elasticity in the general formulation for a sphere with a concentric spherical inclusion has been obtained using the conventional Fourier method. In the scientific works of the classics of natural science of the 19th and 20th centuries, M. G. Lame, W. Thomson, C. Somigliana, V. Cerruti, B. G. Galerkin, G. Fichera, A. I. Lurie, E. Strenberg, and A. F. Ulitko, elastic problems for a solid sphere, space with a spherical cavity, and a sphere with a concentric spherical cavity were solved in various formulations. But even these problems were not strictly justified. The problem considered in this report is much more complex, since it is associated with the conjugation of displacement and stress fields at the inclusion boundary. That's probably why it wasn't considered before. The justification for solving such a problem and establishing its solvability class using the usual Fourier method is based on the analysis of a solvable algebraic system of the sixth order with coefficients that depend on five independent continuous parameters and one discrete one. The general solution of the problem is given in the form of series in terms of axisymmetric vector basis solutions of the Lamé equation for a sphere, constructed by the authors in one of the previous articles. After transitioning to stresses and satisfying the boundary conditions, a resolving system of the above form is obtained. When analyzing the system, a lower estimate of the modulus of its determinant was first found, from which follows not only the unique solvability of the system, but also estimates of the solutions of the system itself. In estimating the determinant, a new classical inequality was proven for one continuous and one discrete parameter, previously unknown to the authors. The next step was to prove a theorem about the conditions that must be imposed on the vector of the external load applied to the surface of the sphere, which ensure the existence of a solution to the problem in a certain class of functions. In the numerical implementation of the solution to the problem, two types of loads on the outer surface of the sphere were considered, which satisfy the equilibrium condition. A computer experiment was conducted with three types of materials for a ball and an inclusion: steel, brass and aluminum. Graphs of normal and tangential stresses on the surface of the inclusion were obtained, and their parametric analysis was performed depending on the geometric and mechanical parameters. The practical convergence of the method was investigated.

Key words: elastic sphere, spherical inclusion, Fourier method, second axisymmetric boundary value problem, exact analytically justified solution, field conjugation, resolving system, solvability class, external load vector, parametric analysis, practical convergence.

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КЛАСИЧНА ЗАДАЧА ПРО ПРУЖНУ КУЛЮ ЗІ СФЕРИЧНИМ ВКЛЮЧЕННЯМ

Уперше звичайним методом Фур'є отримано точний аналітичний обґрунтований розв'язок другої осесиметричної крайової задачі теорії пружності в загальній постановці для кулі з концентричним сферичним включенням. У наукових працях класиків природознавства 19 і 20 століть M. G. Lame, W. Thomson, C. Somigliana, V. Cerruti, B. G. Galerkin, G. Fichera, A. I. Lurie, E. Strenberg, А. Ф. Улітко розв'язувалися пружні задачі для суцільної кулі, простору зі сферичною порожниною і кулі з концентричною сферичною порожниною в різних постановках. Але навіть ці задачі не було строго обґрунтовано. Задача, яка розглядається в цій статті, значно складніша, оскільки пов'язана зі спряженням полів переміщень і напружень на межі включення. Тому, мабуть, її раніше не розглядали. Обґрунтування розв'язку подібної задачі та встановлення її класу розв'язності звичайним методом Фур'є базується на аналізі розв'язувальної алгебраїчної системи шостого порядку з коефіцієнтами, які залежать від п'яти незалежних неперервних параметрів і одного дискретного. Загальний розв'язок задачі подається у вигляді рядів за осесиметричними векторними базисними розв'язками рівняння Ламе для кулі, побудованими авторами в одній з попередніх статей. Після переходу до напружень і задоволення граничних умов отримано розв'язувальну систему вказаного вище вигляду. При аналізі системи вперше знайдено нижню оцінку модуля її визначника, з якої не тільки випливає умова однозначної розв'язності системи, а ще й оцінки розв'язків самої системи. При оцінці визначника було доведено нову класичну нерівність для одного неперервного і одного дискретного параметрів, невідому авторам. Наступним кроком було доведено теорему про умови, які треба накласти на вектор зовнішнього навантаження, прикладеного до поверхні кулі, які забезпечують існування розв'язку задачі в певному класі функції. При чисельній реалізації розв'язку задачі розглядалися два типи навантажень на зовнішню поверхню кулі, які задовольняють умову врівноваженості. Проведено комп'ютерний експеримент з трьома матеріалами кулі і включення: сталь, латунь, алюміній. Отримано графіки нормальних і дотичних напружень на поверхні включення, проведено їх параметричний аналіз в залежності від геометричних і механічних параметрів. Досліджено практичну збіжність методу.

Ключові слова: пружна куля, сферичне включення, метод Фур'є, друга осесиметрична крайова задача, аналітичний обґрунтований розв'язок, спряження полів, розв'язувальна система, клас розв'язності, вектор зовнішнього навантаження, параметричний аналіз, практична збіжність.

Introduction. It can already be said today that new types of materials, among which, first of all, composite and porous materials should be highlighted, have revolutionized industries around the world, offering unprecedented opportunities for innovation. According to expert forecasts, the global market for composite materials alone will grow by an average of 7.2 % each year over the next 5 years. Microelectronics, construction, mechanical and aircraft engineering, renewable energy, chemical industry, rocket and space technology, biology, medicine - this is far from a complete list of areas where various products made of porous and composite materials are used, which, depending on the constituent components, microstructure, size and shape of pores and inclusions, and production technology, have different physical and mechanical properties. Modern technologies place particular emphasis on the creation of nanomaterials with predefined properties. This process is always preceded by mathematical modeling of their possible properties, in particular, strength, which is based on the study of the stress-strain state in the body near inhomogeneities. In this context, the sphere or spherical cavity plays a fundamental role as the most natural and simple heterogeneity in a body. As early as 1933, *J. Goodier* used spherical inhomogeneity in an elastic body to model the stressed state of a material with small air bubbles and steel with slag balls. It is interesting that while there is a sufficient number of scientific articles from the mid-19th century to the present day devoted to problems about an elastic sphere or a space with a spherical cavity, there are no studies of even axisymmetric problems in the general formulation about a sphere with a spherical inclusion or a space with spherical inclusions and a layer. The latter is due to the serious complexity of such problems, especially when strictly justifying their solutions. The question of constructing a justified solution (except for problems for a simple stressed state, where this is not necessary at all) for any problems about a sphere has never been considered at all.

In this work, an exact, well-founded analytical solution of an axisymmetric problem in the general formulation about a loaded elastic ball with a spherical inclusion is obtained for the first time. The conditions for the solvability of this problem by the Fourier method in a certain class of functions are established. The results of a computer experiment with different types of ball and inclusion materials are presented. A parametric analysis of the stresses on the inclusion surface is performed.

Review of previous research results. The problem of an elastic hollow sphere was first considered by *G. Lamé* in his lectures [1]. In this treatise, Lamé considered two problems: on the vibration of a solid sphere and on the normal pressure on the concentric boundary surfaces of a hollow sphere. The work presents the system of Lamé equations in spherical coordinates, introduces a potential function, thanks to which the system of Lamé equations is reduced to a wave equation. Then, the method of separation of variables is used. When solving the Legendre differential equation, some analogs of functions were used, which are today called associated Legendre functions. There is no justification for the solutions given. A closed-form solution was found for the empty sphere. Chronologically, the second work on the empty sphere was done by *W. Thomson* [2] more than 10 years after the publication of Lamé's lectures. Here the problem has already been considered in a general formulation. The solution was written as a power series in the radial coordinate, the coefficients of which are combinations of surface spherical harmonics (for some reason expressed in Cartesian coordinates). After substitution into the equation of equilibrium in the displacements, a system of differential equations was obtained with respect to these harmonics on the surfaces of the sphere. The problems of convergence of the obtained series and justification of the solutions are not raised. In the work of *Somigliana C.* [3], problems for a solid sphere, a space with a spherical cavity, and a sphere with a cavity (displacements are given on concentric surfaces) were considered. Displacements were written by combinations of volume and surface potentials. After that, the problems were broken down into a series of simpler problems for which the integrals with potentials can be expressed in terms of one-dimensional integrals. In the empty sphere problem, the solutions are written as series, about which in the paper it is said: «if convergent, all the conditions of the problem are satisfied. The article [4] is devoted to the solution of the problem for an empty sphere with symmetrical loading on its surfaces. The displacement is written in terms of surface potentials expressed by series similar to the series, which sets the generating function for the Legendre polynomials. The issue of convergence of series is not discussed in the paper. In various formulations, the problem of elastic solid and hollow spheres is considered in the article [5] and the monograph [6]. The solutions are constructed in the form of series by spherical functions. There is no study of the convergence of the obtained series in the work. In [7], an elastic sphere subjected to concentrated forces is considered for the first time. The solution was divided into two groups of terms. The first of them specifies solutions that have a singularity, obtained by the limit transition from a uniformly distributed load in the vicinity of the poles of the sphere. The second group is ordinary solutions in the form of series for spherical functions. The issues of substantiation of the obtained solutions were not considered. In the monograph [8], a similar problem was solved by the method of eigen vector functions. The problem for a spherical inclusion in an unbounded elastic body was probably first considered in [9]. Solutions in the form of series in terms of spherical functions were used to analyze the influence of a small spherical inclusion on the perturbation of a homogeneous stress state in the body.

Let us dwell on the current state of the problem of an elastic sphere with various complications or space with spherical heterogeneity. In [10], simple types of loading of an infinite matrix with spherical inhomogeneity are considered – constant uniaxial and omnidirectional loading. An inclusion or a cavity is chosen as the inhomogeneity. The solutions to the problems are obtained by elementary methods. The displacement of the points of the surface of the inhomogeneity is given by the displacement of the points of the poles and the equator, which are directly related to the external load. The relationship between the displacements of the specified points and the stresses on the boundary of the inhomogeneity is carried out by the method of compatibility of deformations for statically indeterminate systems. In [11], the problem of displacement and rotation of a weakly deformed spherical rigid inclusion embedded in an unbounded elastic medium is considered. The surface of the inclusion is described as a perturbed spherical surface using a term that has the first order of a small parameter. The boundary condition for the displacement vector is set on the perturbed surface and includes the translation vector and the rotation vector. Series in tensor spherical functions and asymptotic series in a small parameter are used. The problem is divided into two separate problems for translations and rotations. The work [12] is devoted to the analysis of the influence of interfacial stresses on the elastic field inside a nanoscale inclusion. The problem is considered in a symmetric formulation. Goodier's approach is used to construct solutions to the Lamé equation in terms of two volumetric spherical functions. The displacement field in the matrix and inclusion is constructed explicitly, provided that the auxiliary harmonic functions are chosen in the simplest form – one or two harmonics. A similar approach was used in the article [13] to solve the problem of the stressed state of an elastic medium with a small spherical cavity. In [14], a closed-form solution is given for the stress fields around a rigid spherical nanoparticle under uniaxial tensile loading. The work explicitly takes into account the presence of an interfacial surface around the nanoparticle with a thickness comparable to the particle size and elastic properties different from those of the matrix. Only the principal terms of displacements and stresses in the matrix and the interfacial spherical layer are taken into account. In [15], the stress field in an infinite body with a spherical inclusion surrounded by a spherical ring embedded in an unbounded matrix phase is investigated. The whole body is subjected to a uniform load in the far field. The approach used is to directly solve the Lamé equation, separately for the deviatoric part and the hydrostatic part of the far field deformations. The displacement is assumed to have a form directly proportional to the far-field deformation, with unknown functions that remain to be determined. Differential equations were obtained for the functions, which were solved analytically. The results were obtained in closed form. Studies of the stressed state of elastic space with a multicomponent system of spherical cavities or inclusions were investigated by the generalized Fourier method in articles [16 – 18]. More complete research results are presented in the monograph [19].

The above bibliographic review shows that the problem of symmetric loading of an elastic sphere with spherical inhomogeneity in a general formulation has not been considered, and the questions of justifying the obtained solutions of other problems for a sphere have not been posed at all.

General formulation of the problem. Consider a sphere centered at a point O of radius R_0, which has a concentric spherical inclusion of radius R_1 ($R_1 < R_0$) made of another material. Let us introduce a spherical coordinate system (r, θ, φ), the origin of which will be aligned with the point O. Let's mark the domains $\Omega_0 = \{(r, \theta, \varphi) : R_1 < r < R_0\}$, $\Omega_1 = \{(r, \theta, \varphi) : r < R_1\}$. We will assume that the material of the part of the sphere that occupies the domain Ω_j has mechanical characteristics (G_j, ν_j) ($j = 0 \div 1$), where G – shear modulus, ν – *Poisson's ratio*, the conditions of ideal mechanical contact are met on the inclusion surface.

Let us consider an axisymmetric boundary value problem in stresses for a piecewise homogeneous sphere $\Omega_0 \bigcup \overline{\Omega_1}$, which is specified by the conditions:

$$\vec{\nabla}^2 \vec{U}_j + \frac{1}{1-2\nu_j} \vec{\nabla}(\vec{\nabla} \vec{U}_j) = 0, \ \vec{x} \in \Omega_j, \ \ j = 0 \div 1; \tag{1}$$

$$(\vec{U}_0)_{|r=R_1} = (\vec{U}_1)_{|r=R_1}, \ \ (F\vec{U}_0)_{|r=R_1} = (F\vec{U}_1)_{|r=R_1}; \tag{2}$$

$$(F\vec{U}_0)_{|r=R_0} = 2G_0 \sum_{n=0}^{\infty} [f_n^{(r)} P_n(\cos\theta) \vec{e}_r + f_n^{(\theta)} P_n^1(\cos\theta) \vec{e}_\theta]. \tag{3}$$

Here $\vec{U}_j$ ($j = 0 \div 1$) denotes the displacement field in the domain Ω_j, $F\vec{U}_j$ is the stress vector on the surface $\Gamma_j = \{(r, \theta, \varphi) : r = R_j\}$ with the normal $\vec{n}_j = \vec{e}_r$, corresponding to the displacement vector $\vec{U}_j$, $\vec{\nabla}$ is the nabla operator, $\{\vec{e}_r, \vec{e}_\theta\}$ is the unit vectors of the spherical coordinate system, $\vec{x}$ is a point in three-dimensional space whose Cartesian coordinates are related to the spherical coordinates (r, θ, φ).

Reducing the problem to a resolving system. Let us solve the boundary value problem (1) – (3) by the usual Fourier method. We will use the results of work [20]. The general solution of equation (1) in the domain Ω_j ($j = 0 \div 1$) can be written as follows:

$$\vec{U}_0(\vec{x}) = \sum_{n=0}^{\infty} [a_{1,n}^{(0)} R_0^{-n+2} \vec{W}_{1,n}^-(r, \theta) + a_{2,n}^{(0)} R_0^{-n} \vec{W}_{2,n}^-(r, \theta)] + \sum_{n=0}^{\infty} [a_{1,n}^{(1)} R_1^{n+3} \vec{W}_{1,n}^+(r, \theta) + a_{2,n}^{(1)} R_1^{n+1} \vec{W}_{2,n}^+(r, \theta)], \ \vec{x} \in \Omega_0, \tag{4}$$

$$\vec{U}_1(\vec{x}) = \sum_{n=0}^{\infty} [b_{1,n} R_1^{-n+2} \vec{W}_{1,n}^-(r, \theta) + b_{2,n} R_1^{-n} \vec{W}_{2,n}^-(r, \theta)], \ \vec{x} \in \Omega_1, \tag{5}$$

where

$$\vec{W}_{1,n}^{\pm}(r, \theta) = \vec{\nabla} w_n^{\pm}(r, \theta), \ \vec{W}_{2,n}^{\pm}(r, \theta) = \chi_n^{\pm} \vec{V}_n^{\pm}(r, \theta) - \zeta_n^{\pm} r^2 \vec{W}_{1,n}^{\pm}(r, \theta), \ \vec{V}_n^{\pm}(r, \theta) = \vec{\nabla}[r^2 w_n^{\pm}(r, \theta)], \tag{6}$$

$$w_n^+(r, \theta) = r^{-n-1} P_n(\cos\theta), \ \ w_n^-(r, \theta) = r^n P_n(\cos\theta),$$

$$\chi_n^+ = n(4\nu - 3) + 2\nu - 2, \ \ \chi_n^- = n(4\nu - 3) + 2\nu - 1, \ \zeta_n^+ = (2n-1)(2\nu - 2), \ \ \zeta_n^- = (2n+3)(2\nu - 2).$$

Here, the vector functions $\{\vec{W}_{1,0}^+(r, \theta), \vec{W}_{1,n}^+(r, \theta), \vec{W}_{2,n}^+(r, \theta)\}_{n=1}^{\infty}$ ($\{\vec{W}_{2,0}^-(r, \theta), \vec{W}_{1,n}^-(r, \theta), \vec{W}_{2,n}^-(r, \theta)\}_{n=1}^{\infty}$) form axisymmetric basis systems of solutions of the Lamé equation for the exterior (interior) of the sphere [20], $P_n(x)$ are *Legendre polynomials*, and $\{a_{i,n}^{(j)}\}_{n=0,i=1}^{\infty,2}$, $j = 0,1$; $\{b_{i,n}\}_{n=0,i=1}^{\infty,2}$ are unknown coefficients. Let us use the formulas obtained in [20] to represent the vectors $\vec{U}_j(\vec{x})$ and $F\vec{U}_j(\vec{x})$ in the form of an expansion over a spherical basis

$$\vec{U}_0(\vec{x}) = \sum_{n=0}^{\infty} a_{1,n}^{(0)} \frac{r^{n-1}}{R_0^{n-2}} [n P_n(\cos\theta) \vec{e}_r + P_n^1(\cos\theta) \vec{e}_\theta] + \sum_{n=0}^{\infty} a_{2,n}^{(0)} \frac{r^{n+1}}{R_0^n} [\beta_{1,n}^{-(0)} P_n(\cos\theta) \vec{e}_r + \beta_{2,n}^{-(0)} P_n^1(\cos\theta) \vec{e}_\theta] +$$

$$+\sum_{n=0}^{\infty} a_{1,n}^{(1)} \frac{R_1^{n+3}}{r^{n+2}}[-(n+1)P_n(\cos\theta)\vec{e}_r + P_n^1(\cos\theta)\vec{e}_\theta] + \sum_{n=0}^{\infty} a_{2,n}^{(1)} \frac{R_1^{n+1}}{r^n}[\beta_{1,n}^{+(0)} P_n(\cos\theta)\vec{e}_r + \beta_{2,n}^{+(0)} P_n^1(\cos\theta)\vec{e}_\theta], \quad (7)$$

$$\vec{U}_1(\vec{x}) = \sum_{n=0}^{\infty} b_{1,n} \frac{r^{n-1}}{R_1^{n-2}}[nP_n(\cos\theta)\vec{e}_r + P_n^1(\cos\theta)\vec{e}_\theta] + \sum_{n=0}^{\infty} b_{2,n} \frac{r^{n+1}}{R_1^n}[\beta_{1,n}^{-(1)} P_n(\cos\theta)\vec{e}_r + \beta_{2,n}^{-(1)} P_n^1(\cos\theta)\vec{e}_\theta], \quad (8)$$

$$\frac{F\vec{U}_0}{2G_0} = \sum_{n=0}^{\infty} a_{1,n}^{(0)} \frac{r^{n-2}}{R_0^{n-2}}(n-1)[nP_n(\cos\theta)\vec{e}_r + P_n^1(\cos\theta)\vec{e}_\theta] + \sum_{n=0}^{\infty} a_{2,n}^{(0)} \frac{r^n}{R_0^n}[\rho_{1,n}^{-(0)} P_n(\cos\theta)\vec{e}_r + \rho_{2,n}^{-(0)} P_n^1(\cos\theta)\vec{e}_\theta] +$$

$$+\sum_{n=0}^{\infty} a_{1,n}^{(1)} \frac{R_1^{n+3}}{r^{n+3}}(n+2)[(n+1)P_n(\cos\theta)\vec{e}_r - P_n^1(\cos\theta)\vec{e}_\theta] + \sum_{n=0}^{\infty} a_{2,n}^{(1)} \frac{R_1^{n+1}}{r^{n+1}}[\rho_{1,n}^{+(0)} P_n(\cos\theta)\vec{e}_r + \rho_{2,n}^{+(0)} P_n^1(\cos\theta)\vec{e}_\theta], \quad (9)$$

$$\frac{F\vec{U}_1}{2G_1} = \sum_{n=0}^{\infty} b_{1,n} \frac{r^{n-2}}{R_1^{n-2}}(n-1)[nP_n(\cos\theta)\vec{e}_r + P_n^1(\cos\theta)\vec{e}_\theta] + \sum_{n=0}^{\infty} b_{2,n} \frac{r^n}{R_1^n}[\rho_{1,n}^{-(1)} P_n(\cos\theta)\vec{e}_r + \rho_{2,n}^{-(1)} P_n^1(\cos\theta)\vec{e}_\theta], \quad (10)$$

where

$$\beta_{1,n}^{+(j)} = -n(n+3-4\nu_j), \quad \beta_{2,n}^{+(j)} = n+4\nu_j-4, \quad \beta_{1,n}^{-(j)} = (n+1)(n+4\nu_j-2), \quad \beta_{2,n}^{-(j)} = n-4\nu_j+5,$$

$$\rho_{1,n}^{+(j)} = n(n^2+3n-2\nu_j), \quad \rho_{2,n}^{+(j)} = -(n^2+2\nu_j-2), \quad \rho_{1,n}^{-(j)} = (n+1)(n^2-n-2\nu_j-2), \quad \rho_{2,n}^{-(j)} = n^2+2n+2\nu_j-1.$$

Let us substitute formulas (7) – (10) into the conjugation conditions (2) and the boundary condition (3). After equating the coefficients in the corresponding Legendre functions, we arrive at the algebraic system with respect to the unknowns $\{a_{i,n}^{(j)}\}_{n=0,i=1}^{\infty,2}$, $j=0,1$; $\{b_{i,n}\}_{n=0,i=1}^{\infty,2}$

$$n\left(\frac{R_1}{R_0}\right)^{n-2} a_{1,n}^{(0)} + (n+1)(n+4\nu_0-2)\left(\frac{R_1}{R_0}\right)^n a_{2,n}^{(0)} - (n+1)a_{1,n}^{(1)} - n(n+3-4\nu_0)a_{2,n}^{(1)} = nb_{1,n} + (n+1)(n+4\nu_1-2)b_{2,n}, \quad (11)$$

$$\left(\frac{R_1}{R_0}\right)^{n-2} a_{1,n}^{(0)} + (n-4\nu_0+5)\left(\frac{R_1}{R_0}\right)^n a_{2,n}^{(0)} + a_{1,n}^{(1)} + (n+4\nu_0-4)a_{2,n}^{(1)} = b_{1,n} + (n-4\nu_1+5)b_{2,n}, \quad (12)$$

$$(n+1)(n+2)a_{1,n}^{(1)} + n(n^2+3n-2\nu_0)a_{2,n}^{(1)} + n(n-1)\left(\frac{R_1}{R_0}\right)^{n-2} a_{1,n}^{(0)} + (n+1)(n^2-n-2\nu_0-2)\left(\frac{R_1}{R_0}\right)^n a_{2,n}^{(0)} =$$

$$= \frac{G_1}{G_0}[n(n-1)b_{1,n} + (n+1)(n^2-n-2\nu_1-2)b_{2,n}], \quad (13)$$

$$-(n+2)a_{1,n}^{(1)} - (n^2+2\nu_0-2)a_{2,n}^{(1)} + (n-1)\left(\frac{R_1}{R_0}\right)^{n-2} a_{1,n}^{(0)} + (n^2+2n+2\nu_0-1)\left(\frac{R_1}{R_0}\right)^n a_{2,n}^{(0)} =$$

$$= \frac{G_1}{G_0}[(n-1)b_{1,n} + (n^2+2n+2\nu_1-1)b_{2,n}], \quad (14)$$

$$n(n-1)a_{1,n}^{(0)} + (n+1)(n^2-n-2\nu_0-2)a_{2,n}^{(0)} + (n+1)(n+2)\left(\frac{R_1}{R_0}\right)^{n+3} a_{1,n}^{(1)} + n(n^2+3n-2\nu_0)\left(\frac{R_1}{R_0}\right)^{n+1} a_{2,n}^{(1)} = f_n^{(r)}, \quad (15)$$

$$(n-1)a_{1,n}^{(0)} + (n^2+2n+2\nu_0-1)a_{2,n}^{(0)} - (n+2)\left(\frac{R_1}{R_0}\right)^{n+3} a_{1,n}^{(1)} - (n^2+2\nu_0-2)\left(\frac{R_1}{R_0}\right)^{n+1} a_{2,n}^{(1)} = f_n^{(\theta)}. \quad (16)$$

From the first two equations of the resulting system, we find $b_{1,n}, b_{2,n}$

$$b_{1,n} = \left[1 - \frac{(2n+1)}{\Delta_n^{-(1)}}\beta_{2,n}^{-(1)}\right]a_{1,n}^{(1)} + \left[\beta_{2,n}^{+(0)} - \frac{n(2n-1)}{\Delta_n^{-(1)}}\beta_{2,n}^{-(1)}\right]a_{2,n}^{(1)} + \left(\frac{R_1}{R_0}\right)^{n-2} a_{1,n}^{(0)} + \left[\beta_{2,n}^{-(0)} - \frac{\Delta_n^{-(0)}}{\Delta_n^{-(1)}}\beta_{2,n}^{-(1)}\right]\left(\frac{R_1}{R_0}\right)^n a_{2,n}^{(0)}, \quad (17)$$

$$b_{2,n} = \frac{(2n+1)}{\Delta_n^{-(1)}}a_{1,n}^{(1)} + \frac{n(2n-1)}{\Delta_n^{-(1)}}a_{2,n}^{(1)} + \frac{\Delta_n^{-(0)}}{\Delta_n^{-(1)}}\left(\frac{R_1}{R_0}\right)^n a_{2,n}^{(0)}, \quad (18)$$

where $\Delta_n^{-(j)} = 2[(3-4\nu_j)n + 1 - 2\nu_j] > 0$.

Let us eliminate the unknowns $b_{1,n}, b_{2,n}$ from equations (13) – (16). After some transformations, we write

$$\left\{2[n^2 + (1-2\nu_0)n + 1 - \nu_0] + 2\frac{G_1}{G_0}(n-1)[(3-4\nu_0)n + 2 - 2\nu_0]\right\}a_{2,n}^{(1)} +$$

$$+\left(1-\frac{G_1}{G_0}\right)(n-1)(2n+1)\left(\frac{R_1}{R_0}\right)^{n-2}a_{1,n}^{(0)}+\left(1-\frac{G_1}{G_0}\right)(n-1)(n+1)(2n+3)\left(\frac{R_1}{R_0}\right)^{n}a_{2,n}^{(0)}=0\,, \qquad (19)$$

$$\left[n+2+2\frac{G_1}{G_0}\frac{[n^2+(1+2\nu_1)n+1+\nu_1]}{\Delta_n^{-(1)}}\right]\left[(2n+1)a_{1,n}^{(1)}+n(2n-1)a_{2,n}^{(1)}\right]+$$

$$+\left[2\frac{G_1}{G_0}[n^2+(1+2\nu_1)n+1+\nu_1]\frac{\Delta_n^{-(0)}}{\Delta_n^{-(1)}}-2[n^2+(1+2\nu_0)n+1+\nu_0]\right]\left(\frac{R_1}{R_0}\right)^{n}a_{2,n}^{(0)}=0\,. \qquad (20)$$

$$(n-1)(2n+1)a_{1,n}^{(0)}+(n-1)(n+1)(2n+3)a_{2,n}^{(0)}+2[n^2+(1-2\nu_0)n+1-\nu_0]\left(\frac{R_1}{R_0}\right)^{n+1}a_{2,n}^{(1)}=f_n^{(r)}+(n+1)f_n^{(\theta)}\,, \qquad (21)$$

$$-2[n^2+(1+2\nu_0)n+1+\nu_0]a_{2,n}^{(0)}+(n+2)(2n+1)\left(\frac{R_1}{R_0}\right)^{n+3}a_{1,n}^{(1)}+n(n+2)(2n-1)\left(\frac{R_1}{R_0}\right)^{n+1}a_{2,n}^{(1)}=f_n^{(r)}-nf_n^{(\theta)}\,. \qquad (22)$$

Analysis of the resolving system. Let us write the determinant of the resolving linear algebraic system (19) – (22) (the order of the rows of the determinant is given from the last equation to the first)

$$\Delta_n=\begin{vmatrix}(n+2)(2n+1)\rho^{n+3} & n(n+2)(2n-1)\rho^{n+1} & 0 & -d_n^{+(0)}\\ 0 & d_n^{-(0)}\rho^{n+1} & \alpha_n & \beta_n\\ (2n+1)\Delta_n^{(3)} & n(2n-1)\Delta_n^{(3)} & 0 & \Delta_n^{(4)}\rho^{n}\\ 0 & \Delta_n^{(2)} & (1-G_{10})\alpha_n\rho^{n-2} & (1-G_{10})\beta_n\rho^{n}\end{vmatrix}\,, \qquad (23)$$

where

$$d_n^{\pm(j)}=2[n^2+(1\pm 2\nu_j)n+1\pm\nu_j],\ \ \rho=\frac{R_1}{R_0},\ \ G_{10}=\frac{G_1}{G_0}\,,\ \ \Delta_n^{(2)}=d_n^{-(0)}+2G_{10}(n-1)[(3-4\nu_0)n+2-2\nu_0]\,,$$

$$\Delta_n^{(3)}=n+2+G_{10}\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}\,,\ \ \Delta_n^{(4)}=G_{10}\frac{\Delta_n^{-(0)}d_n^{+(1)}}{\Delta_n^{-(1)}}-d_n^{+(0)}\,,\ \ \alpha_n=(n-1)(2n+1),\ \ \beta_n=(n-1)(n+1)(2n+3)\,.$$

Let us expand the determinant (23) and write it as an expansion in powers of the variable G_{10}

$$\Delta_n=I_n^{(0)}+I_n^{(1)}G_{10}+I_n^{(2)}G_{10}^2\,, \qquad (24)$$

where

$$\begin{aligned}I_n^{(0)}=&-\alpha_n n(n+2)^2(2n-1)(2n+1)\beta_n\rho^{2n+3}-\alpha_n(n+2)(2n+1)d_n^{-(0)}d_n^{+(0)}\rho^{2n+3}+\\&+\alpha_n\beta_n n(n+2)^2(2n-1)(2n+1)\rho^{2n+1}+\alpha_n(n+2)(2n+1)d_n^{+(0)}d_n^{-(0)}+\\&+\alpha_n(n+2)(2n+1)d_n^{+(0)}d_n^{-(0)}\rho^{4n+2}+\alpha_n n(2n-1)(n+2)^2(2n+1)\beta_n\rho^{2n+1}-\\&-\alpha_n n(n+2)^2(2n-1)(2n+1)\beta_n\rho^{2n-1}-\alpha_n(n+2)(2n+1)d_n^{+(0)}d_n^{-(0)}\rho^{2n-1}\,,\end{aligned} \qquad (25)$$

$$\begin{aligned}I_n^{(1)}=&\,\alpha_n n(n+2)^2(2n-1)(2n+1)\beta_n\rho^{2n+3}-\alpha_n n(n+2)(2n-1)(2n+1)\beta_n\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}\rho^{2n+3}+\\&+\alpha_n(n+2)(2n+1)d_n^{-(0)}\frac{\Delta_n^{-(0)}}{\Delta_n^{-(1)}}d_n^{+(1)}\rho^{2n+3}-\alpha_n(n+2)(2n+1)(n-1)(\Delta_n^{-(0)}+2)d_n^{+(0)}\rho^{2n+3}-\\&-\alpha_n\beta_n n(n+2)^2(2n-1)(2n+1)\rho^{2n+1}+\alpha_n\beta_n n(n+2)(2n-1)(2n+1)\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}\rho^{2n+1}+\\&+\alpha_n(2n+1)d_n^{+(0)}d_n^{-(0)}\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}+\alpha_n(n+2)(2n+1)d_n^{+(0)}(n-1)(\Delta_n^{-(0)}+2)-\\&-\alpha_n(n+2)(2n+1)d_n^{+(0)}d_n^{-(0)}\rho^{4n+2}-\alpha_n(n+2)(2n+1)\frac{\Delta_n^{-(0)}}{\Delta_n^{-(1)}}d_n^{+(1)}d_n^{-(0)}\rho^{4n+2}-\\&-\alpha_n n(2n-1)(n+2)^2(2n+1)\beta_n\rho^{2n+1}+\alpha_n n(2n-1)(n+2)(2n+1)\beta_n\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}\rho^{2n+1}-\\&+\alpha_n n(n+2)^2(2n-1)(2n+1)\beta_n\rho^{2n-1}-\alpha_n n(n+2)(2n-1)(2n+1)\beta_n\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}\rho^{2n-1}+\end{aligned}$$

$$+\alpha_n(n+2)(2n+1)d_n^{+(0)}d_n^{-(0)}\rho^{2n-1}-\alpha_n(2n+1)d_n^{+(0)}d_n^{-(0)}\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}\rho^{2n-1}, \tag{26}$$

$$I_n^{(2)}=\alpha_n n(n+2)(2n-1)(2n+1)\beta_n\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}\rho^{2n+3}+\alpha_n(n-1)(n+2)(2n+1)(\Delta_n^{-(0)}+2)\frac{\Delta_n^{-(0)}}{\Delta_n^{-(1)}}d_n^{+(1)}\rho^{2n+3}-$$

$$-\alpha_n\beta_n n(n+2)(2n-1)(2n+1)\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}\rho^{2n+1}+\alpha_n(n-1)(2n+1)d_n^{+(0)}(\Delta_n^{-(0)}+2)]\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}+$$

$$+\alpha_n(n+2)(2n+1)\frac{\Delta_n^{-(0)}}{\Delta_n^{-(1)}}d_n^{+(1)}d_n^{-(0)}\rho^{4n+2}-\alpha_n n(2n-1)(n+2)(2n+1)\beta_n\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}\rho^{2n+1}+$$

$$+\alpha_n n(n+2)(2n-1)(2n+1)\beta_n\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}\rho^{2n-1}+\alpha_n(2n+1)d_n^{+(0)}d_n^{-(0)}\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}\rho^{2n-1}. \tag{27}$$

Theorem 1. *The multiparameter determinant* Δ_n (23) *for all values of the parameters* $G_i>0,\ \nu_i\in[-1;0,5)$ $(i=0,1)$, $\rho\in(0,1)$, *and an arbitrary natural number* $n\ge 2$ *is positive. Moreover, the inequality holds*

$$\Delta_n > G_{10}^2(n^2-1)^3 n(2n+1). \tag{28}$$

To prove the theorem, we first prove a new classical inequality.

Lemma 1. *For* $\rho\in[0,1]$ *and an arbitrary natural number* n, *the classical inequality holds*

$$4(1-\rho^{2n-1}-\rho^{2n+3}+\rho^{4n+2})\ge(2n-1)(2n+3)(\rho^{2n-1}-2\rho^{2n+1}+\rho^{2n+3}). \tag{29}$$

Proof of the lemma. Let's factorize the expression

$$4(1-\rho^{2n-1}-\rho^{2n+3}+\rho^{4n+2})-(2n-1)(2n+3)(\rho^{2n-1}-2\rho^{2n+1}+\rho^{2n+3})=$$

$$=(1-\rho)^2\left[4\sum_{i=0}^{2n-2}\rho^i\sum_{k=0}^{2k+2}\rho^k-(2n-1)(2n+3)\rho^{2n-1}(1+\rho)^2\right].$$

Let us prove that the expression in square brackets is non-negative. Since $(1+\rho)^2\le 2(1+\rho^2)$, it is actually sufficient to prove the inequality

$$2\sum_{i=0}^{2n-2}\rho^i\sum_{k=0}^{2k+2}\rho^k-(2n-1)(2n+3)\rho^{2n-1}(1+\rho^2)\ge 0. \tag{30}$$

By replacing the summation indices in the product of series, it can be represented as

$$\sum_{i=0}^{2n-2}\rho^i\sum_{k=0}^{2k+2}\rho^k=\sum_{m=0}^{2n-2}(m+1)\rho^m+(2n-1)\sum_{m=2n-1}^{2n+2}\rho^m+\sum_{m=2n+3}^{4n}(4n+1-m)\rho^m. \tag{31}$$

We perform the identity transformation of the expression that is included in the left side of the inequality (30)

$$2\sum_{i=0}^{2n-2}\rho^i\sum_{k=0}^{2k+2}\rho^k-(2n-1)(2n+3)\rho^{2n-1}(1+\rho^2)=\sum_{m=1}^{2n-2}m(m+1)\left[\rho^{m-1}(1-\rho)-\rho^{4n-m}(1-\rho)\right]+$$

$$+2n(2n-1)\left[\rho^{2n-2}(1-\rho)-\rho^{2n+1}(1-\rho)\right]-(2n-1)\left[\rho^{2n-1}(1-\rho)-\rho^{2n}(1-\rho)\right]. \tag{32}$$

All expressions in square brackets in (32) are non-negative at $\rho\in[0,1]$. In addition,

$$\left[\rho^{2n-2}(1-\rho)-\rho^{2n+1}(1-\rho)\right]\ge\left[\rho^{2n-1}(1-\rho)-\rho^{2n}(1-\rho)\right].$$

Therefore, the inequality

$$2\sum_{i=0}^{2n-2}\rho^i\sum_{k=0}^{2k+2}\rho^k-(2n-1)(2n+3)\rho^{2n-1}(1+\rho^2)\ge 0, \tag{33}$$

holds, and the lemma is fulfilled with it.

Proof of the theorem. Let us prove that all coefficients $I_n^{(i)}$ $(i=0\div 2)$ in formula (24) are either non-negative or positive for arbitrary values of the parameters specified in the conditions of the theorem. Let us transform $I_n^{(0)}$ to the following form

$$I_n^{(0)}=\alpha_n(n+2)(2n+1)\left[d_n^{+(0)}d_n^{-(0)}(1-\rho^{2n-1}-\rho^{2n+3}+\rho^{4n+2})-n(n+2)(2n-1)\beta_n(\rho^{2n-1}-2\rho^{2n+1}+\rho^{2n+3})\right].$$

Using the result of the lemma, we can estimate

$$I_n^{(0)}\ge\alpha_n(n+2)(2n-1)(2n+1)(2n+3)\left[\frac{1}{4}d_n^{+(0)}d_n^{-(0)}-n(n+2)(n-1)(n+1)\right]\rho^{2n-1}(1-\rho^2)^2.$$

The expression in square brackets can be written as:

$$\frac{1}{4}d_n^{+(0)}d_n^{-(0)} - n(n+2)(n-1)(n+1) = [n^2+(1+2\nu_0)n+1+\nu_0][n^2+(1-2\nu_0)n+1-\nu_0]-(n^2+2n)(n^2-1) .$$

The smallest value of the previous expression as a function of the parameter ν_0 on the interval $\nu_0 \in [-1;0,5)$ is taken at $\nu_0 = -1$ and it is equal to zero. Therefore,

$$I_n^{(0)} \ge 0 . \tag{34}$$

In addition, the inequality

$$\left[d_n^{+(0)}d_n^{-(0)}(1-\rho^{2n-1}-\rho^{2n+3}+\rho^{4n+2})-n(n+2)(2n-1)\beta_n(\rho^{2n-1}-2\rho^{2n+1}+\rho^{2n+3})\right] \ge 0 . \tag{35}$$

is proved.

Now let's convert the coefficient $I_n^{(1)}$

$$I_n^{(1)} = \alpha_n(2n+1)\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}\Big[d_n^{+(0)}d_n^{-(0)}(1-\rho^{2n-1}-\rho^{2n-3}+\rho^{4n+2})-$$

$$-n(n+2)(2n-1)\beta_n(\rho^{2n-1}-2\rho^{2n+1}+\rho^{2n+3})\Big]+\alpha_n(2n+1)\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}d_n^{+(0)}d_n^{-(0)}(\rho^{2n-3}-\rho^{4n+2})+$$

$$+\alpha_n(2n+1)\Bigg[(n-1)(n+2)d_n^{+(0)}(\Delta_n^{-(0)}+2)(1-\rho^{2n+3})+(n+2)d_n^{+(0)}d_n^{-(0)}(\rho^{2n-1}-\rho^{4n+2})+$$

$$+n(n+2)^2(2n-1)\beta_n(\rho^{2n-1}-2\rho^{2n+1}+\rho^{2n+3})+(n+2)\frac{\Delta_n^{-(0)}}{\Delta_n^{-(1)}}d_n^{+(1)}d_n^{-(0)}(\rho^{2n+3}-\rho^{4n+2})\Bigg].$$

Due to inequality (35), the first term of the previous formula is nonnegative, and all the others are positive.

The coefficient $I_n^{(2)}$ can be written as:

$$I_n^{(2)} = \alpha_n n(n+2)(2n-1)(2n+1)\beta_n\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}\rho^{2n-1}(1-\rho^2)^2+\alpha_n(n-1)(n+2)(2n+1)(\Delta_n^{-(0)}+2)\frac{\Delta_n^{-(0)}}{\Delta_n^{-(1)}}d_n^{+(1)}\rho^{2n+3}+$$

$$+\alpha_n(n-1)(2n+1)d_n^{+(0)}(\Delta_n^{-(0)}+2)]\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}+\alpha_n(2n+1)d_n^{+(0)}d_n^{-(0)}\frac{d_n^{+(1)}}{\Delta_n^{-(1)}}\rho^{2n-1}+\alpha_n(n+2)(2n+1)\frac{\Delta_n^{-(0)}}{\Delta_n^{-(1)}}d_n^{+(1)}d_n^{-(0)}\rho^{4n+2} .$$

From this expression of the coefficient $I_n^{(2)}$ it follows that all its terms are positive and, as a consequence,

$$I_n^{(2)} > \alpha_n(2n+1)(n-1)d_n^{+(0)}(\Delta_n^{-(0)}+2)]\frac{d_n^{+(1)}}{\Delta_n^{-(1)}} =$$

$$= 4(n-1)^2(2n+1)^2[n^2+(1+2\nu_0)n+1+\nu_0][(3-4\nu_0)n+2-2\nu_0]\frac{n^2+(1+2\nu_1)n+1+\nu_1}{(3-4\nu_1)n+1-2\nu_1} .$$

The minimum value of the last expression at $\nu_i \in [-1;0,5)$ $(i=0,1)$ is equal to

$$4(n-1)^2(2n+1)^2[n^2+2n+3/2](n+1)\frac{n^2-n}{7n+3} > (n^2-1)^3 n(2n+1) ,$$

which finally proves the theorem.

Theorem 1 makes it possible to reasonably construct an exact solution to problem (1) – (3).

Theorem 2. *If the series*

$$\sum_{n=0}^{\infty}(|f_n^{(r)}|+n|f_n^{(\theta)}|) , \tag{36}$$

converges and external load

$$\vec{f}(\theta) = 2G_0\sum_{n=0}^{\infty}[f_n^{(r)}P_n(\cos\theta)\vec{e}_r + f_n^{(\theta)}P_n^1(\cos\theta)\vec{e}_\theta] ,$$

acting on the surface Γ_0 *of the sphere* $\Omega_0 \cup \overline{\Omega_1}$ *, is balanced*

$$\oiint_{\Gamma_0}(\vec{f}(\theta),\vec{e}_z)dS = 0 , \tag{37}$$

then there exists a unique solution to problem (1) – (3) up to the rigid displacement vector, which has the form

$$\vec{U}(r,\theta)=\begin{cases}\vec{U}_0(r,\theta), & R_1<r<R_0,\\ \vec{U}_1(r,\theta), & 0\le r<R_1\end{cases} \tag{38}$$

and belongs to space $C^2(\Omega_0\bigcup\Omega_1)\bigcap C^1(\Omega_0\bigcup\overline{\Omega_1})\bigcap C(\overline{\Omega_0\bigcup\Omega_1})$. *Vector functions* $\vec{U}_0,\vec{U}_1$ *are given by formulas (4), (5).*

Proof of the theorem. Let us analyze the solution system of equations (19) – (22) separately for $n=0$, $n=1$ and $n\ge 2$.

For $n=0$ there are no equations after equating the coefficients for $P_0^1\equiv 0$, therefore, we have only three equations

$$-a_{1,0}^{(1)}+(4\nu_0-2)a_{2,0}^{(0)}=(4\nu_1-2)b_{2,0}\,, \tag{39}$$

$$2a_{1,0}^{(1)}-(2+2\nu_0)a_{2,0}^{(0)}=-G_{10}(2+2\nu_1)b_{2,0}\,, \tag{40}$$

$$-(2+2\nu_0)a_{2,0}^{(0)}+2\rho^3a_{1,0}^{(1)}=f_0^{(r)}\,. \tag{41}$$

System (39) – (41) has a unique solution

$$a_{2,0}^{(0)}=-\left(1+G_{10}\frac{1+\nu_1}{2-4\nu_1}\right)\frac{f_0^{(r)}}{2\bar{\Delta}_0},\quad a_{1,0}^{(1)}=-(1+\nu_0)\left(1+G_{10}\frac{1+\nu_1}{2-4\nu_1}\right)\frac{f_0^{(r)}}{2\bar{\Delta}_0}\rho^{-3}+\frac{f_0^{(r)}}{2}\rho^{-3}\,, \tag{42}$$

$$b_{2,0}=\frac{1}{2-4\nu_1}\left[-(1+\nu_0)\left(1+G_{10}\frac{1+\nu_1}{2-4\nu_1}\right)\frac{f_0^{(r)}}{2\bar{\Delta}_0}\rho^{-3}+\frac{f_0^{(r)}}{2}\rho^{-3}\right]-\frac{2-4\nu_0}{2-4\nu_1}\left(1+G_{10}\frac{1+\nu_1}{2-4\nu_1}\right)\frac{f_0^{(r)}}{2\bar{\Delta}_0}\,, \tag{41}$$

where

$$\bar{\Delta}_0=(1+\nu_0)(1-\rho^3)+\frac{1+\nu_1}{2-4\nu_1}G_{10}[1+\nu_0+(2-4\nu_0)\rho^3]>0\,.$$

Since the solutions $\vec{W}_{2,0}^{+}(r,\theta)\equiv 0$, $\vec{W}_{1,0}^{-}(r,\theta)\equiv 0$, then the coefficients $a_{2,0}^{(1)}$, $a_{1,0}^{(0)}$ and $b_{1,0}$ can be chosen arbitrarily.

The resolving system for $n=1$ has the form

$$\rho^{-1}a_{1,1}^{(0)}+2(4\nu_0-1)\rho a_{2,1}^{(0)}-2a_{1,1}^{(1)}-(4-4\nu_0)a_{2,1}^{(1)}=b_{1,1}+2(4\nu_1-1)b_{2,1}\,, \tag{42}$$

$$\rho^{-1}a_{1,1}^{(0)}+(6-4\nu_0)\rho a_{2,1}^{(0)}+a_{1,1}^{(1)}+(4\nu_0-3)a_{2,1}^{(1)}=b_{1,1}+(6-4\nu_1)b_{2,1}\,, \tag{43}$$

$$6a_{1,1}^{(1)}+(4-2\nu_0)a_{2,1}^{(1)}-4(1+\nu_0)\rho a_{2,1}^{(0)}=-4G_{10}(1+\nu_1)b_{2,1}\,, \tag{44}$$

$$-3a_{1,1}^{(1)}+(1-2\nu_0)a_{2,1}^{(1)}+2(1+\nu_0)\rho a_{2,1}^{(0)}=2G_{10}(1+\nu_1)b_{2,1}\,, \tag{45}$$

$$6(1-\nu_0)\rho^2a_{2,1}^{(1)}=f_1^{(r)}+2f_1^{(\theta)}\,, \tag{46}$$

$$-2(1+\nu_0)a_{2,1}^{(0)}+3\rho^4a_{1,n}^{(1)}+\rho^2a_{2,1}^{(1)}=(f_1^{(r)}-f_1^{(\theta)})/3\,. \tag{47}$$

Note that the statics condition (37) for the external load leads to the following relation:

$$f_1^{(r)}+2f_1^{(\theta)}=0\,.$$

Then from equation (46) it follows that $a_{2,1}^{(1)}=0$. Two equations (44), (45) are proportional, so one of them must be eliminated. There remains a system of four equations (42), (43), (45), (47) with respect to the unknowns $a_{1,1}^{(1)}$, $a_{1,1}^{(0)}$, $a_{2,1}^{(0)}$, $b_{2,1}$. The variable $b_{1,1}$ is free and can take on any values. The term with it in the general solution corresponds to the rigid displacement vector. We fix the center of the sphere and then $b_{1,1}=0$. After substituting into the previous system $a_{2,1}^{(1)}=0$, $b_{1,1}=0$ and simplifying, we obtain

$$b_{2,1}=\frac{3}{10}\rho^{-1}a_{1,1}^{(0)}+\rho a_{2,1}^{(0)}\,, \tag{48}$$

$$a_{1,1}^{(1)}+\left[1-\frac{3}{10}(6-4\nu_1)\right]\rho^{-1}a_{1,1}^{(0)}+4(\nu_1-\nu_0)\rho a_{2,1}^{(0)}=0\,, \tag{49}$$

$$-3a_{1,1}^{(1)}-\frac{3}{10}G_{10}(2+2\nu_1)\rho^{-1}a_{1,1}^{(0)}+[2+2\nu_0-G_{10}(2+2\nu_1)]\rho a_{2,1}^{(0)}=0\,, \tag{50}$$

$$3\rho^4a_{1,1}^{(1)}-2(1+\nu_0)a_{2,1}^{(0)}=\frac{1}{3}(f_1^{(r)}-f_1^{(\theta)})\,. \tag{51}$$

System (48) – (51) has the determinant

$$\overline{\Delta}_1 = \begin{vmatrix} 1 & \left[1-\frac{3}{10}(6-4\nu_1)\right]\rho^{-1} & 4(\nu_1-\nu_0)\rho \\ -3 & -\frac{3}{10}G_{10}(2+2\nu_1)\rho^{-1} & [2+2\nu_0-G_{10}(2+2\nu_1)]\rho \\ 3\rho^4 & 0 & -2(1+\nu_0) \end{vmatrix} =$$

$$= 6(1+\nu_0)\frac{8-12\nu_1}{10}(\rho^{-1}-\rho^4)+\frac{12}{10}G_{10}(1+\nu_0)(1+\nu_1)\rho^{-1}+6G_{10}\frac{8-12\nu_0}{10}(1+\nu_1)\rho^4 . \quad (52)$$

Formula (52) shows that the determinant $\overline{\Delta}_1$ is positive. Therefore, the system (48) – (51) has a unique solution

$$a_{1,1}^{(1)} = \frac{1}{15\overline{\Delta}_1}(f_1^{(r)}-f_1^{(\theta)})\left[-(8-12\nu_1)(1+\nu_0)+G_{10}(8-12\nu_0)(1+\nu_1)\right], \quad (53)$$

$$a_{1,1}^{(0)} = -\frac{2}{3\overline{\Delta}_1}(f_1^{(r)}-f_1^{(\theta)})[1+6\nu_1-5\nu_0-G_{10}(1+\nu_1)]\rho , \quad (54)$$

$$a_{2,1}^{(0)} = -\frac{1}{5\overline{\Delta}_1}(f_1^{(r)}-f_1^{(\theta)})\left[4-6\nu_1+G_{10}(1+\nu_1)\right]\rho^{-1} , \; b_{2,1} = -\frac{1-\nu_0}{\overline{\Delta}_1}(f_1^{(r)}-f_1^{(\theta)}) . \quad (55)$$

Now consider the resolving system at $n \ge 2$. Since by Theorem 1 the determinant of the system is positive, the system has a unique solution, which looks like this:

$$a_{1,n}^{(1)} = \frac{f_n^{(r)}-nf_n^{(\theta)}}{\Delta_n}\left\{\alpha_n\Delta_n^{(2)}\Delta_n^{(4)}\rho^n+(1-G_{10})[\alpha_n n(2n-1)\beta_n\Delta_n^{(3)}(\rho^{n-2}-\rho^n)-\alpha_n d_n^{-(0)}\Delta_n^{(4)}\rho^{3n-1}]\right\}+$$

$$+\frac{f_n^{(r)}+(n+1)f_n^{(\theta)}}{\Delta_n}(1-G_{10})[\alpha_n n(2n-1)\Delta_n^{(3)}d_n^{+(0)}\rho^{n-2}+\alpha_n n(n+2)(2n-1)\Delta_n^{(4)}\rho^{3n-1}] , \quad (56)$$

$$a_{2,n}^{(1)} = -\frac{f_n^{(r)}-nf_n^{(\theta)}}{\Delta_n}(1-G_{10})\alpha_n(2n+1)\beta_n\Delta_n^{(3)}(\rho^{n-2}-\rho^n)-$$

$$-\frac{f_n^{(r)}+(n+1)f_n^{(\theta)}}{\Delta_n}(1-G_{10})[\alpha_n(2n+1)\Delta_n^{(3)}d_n^{+(0)}\rho^{n-2}+\alpha_n(n+2)(2n+1)\Delta_n^{(4)}\rho^{3n+1}] , \quad (57)$$

$$a_{1,n}^{(0)} = \frac{f_n^{(r)}-nf_n^{(\theta)}}{\Delta_n}[(2n+1)\beta_n\Delta_n^{(2)}\Delta_n^{(3)}-(1-G_{10})(2n+1)\beta_n\Delta_n^{(3)}d_n^{-(0)}\rho^{2n+1}]+$$

$$+\frac{f_n^{(r)}+(n+1)f_n^{(\theta)}}{\Delta_n}\Big[(2n+1)\Delta_n^{(2)}\Delta_n^{(3)}d_n^{+(0)}+(n+2)(2n+1)\Delta_n^{(2)}\Delta_n^{(4)}\rho^{2n+3}+$$

$$+(1-G_{10})n(n+2)(2n-1)(2n+1)\beta_n\Delta_n^{(3)}(\rho^{2n+1}-\rho^{2n+3})\Big] , \quad (58)$$

$$a_{2,n}^{(0)} = -\frac{f_n^{(r)}-nf_n^{(\theta)}}{\Delta_n}[\alpha_n(2n+1)\Delta_n^{(2)}\Delta_n^{(3)}-(1-G_{10})\alpha_n(2n+1)\Delta_n^{(3)}d_n^{-(0)}\rho^{2n-1}]-$$

$$-\frac{f_n^{(r)}+(n+1)f_n^{(\theta)}}{\Delta_n}(1-G_{10})\alpha_n n(n+2)(2n-1)(2n+1)\Delta_n^{(3)}(\rho^{2n-1}-\rho^{2n+1}) . \quad (59)$$

From formulas (56) – (59) and Theorem 1, we obtain estimates of the coefficients of the constructed solutions (4), (5) at $n \ge 2$

$$|a_{1,n}^{(1)}| < C_1(|f_n^{(r)}|+n|f_n^{(\theta)}|)\rho^{n-2} , \; |a_{2,n}^{(1)}| < C_2 n^{-1}(|f_n^{(r)}|+n|f_n^{(\theta)}|)\rho^{n-2} , \; |a_{1,n}^{(0)}| < C_3 n^{-1}(|f_n^{(r)}|+n|f_n^{(\theta)}|) ,$$

$$|a_{2,n}^{(0)}| < C_4 n^{-2}(|f_n^{(r)}|+n|f_n^{(\theta)}|) , \; |b_{1,n}| < C_5 n(|f_n^{(r)}|+n|f_n^{(\theta)}|)\rho^{n-2} , \; |b_{2,n}| < C_6(|f_n^{(r)}|+n|f_n^{(\theta)}|)\rho^{n-2} ,$$

where $(C_k)_{k=1}^6$ are positive constants that do not depend on n. These estimates ensure absolute and uniform convergence of the series in formulas (4), (5) in the sphere up to its boundary, as well as the fulfillment of the condition $\vec{U} \in C^2(\Omega_0 \cup \Omega_1) \cap C^1(\Omega_0 \cup \overline{\Omega_1}) \cap C(\overline{\Omega_0 \cup \Omega_1})$. So, the theorem is proven.

Computer experiment. Problem (1) – (3) was solved numerically under the following conditions: the ball material was chosen to be steel with elastic constants $G_0 = 82$ GPa, $\nu_0 = 0.28$, the inclusion materials were chosen to be brass ($G_1 = 35.2$ GPa, $\nu_1 = 0.35$) or aluminum ($G_1 = 26$ GPa, $\nu_1 = 0.34$).

The first type of load. The external load on the surface of the sphere is given by a vector function $\vec{f}(\theta) = -\sigma \sin\theta \vec{e}_r$ (it is automatically balanced). Fig. 1 shows the distribution of normal stresses on the surface of a brass inclusion depending on the relative size of the inclusion $\rho = R_1 / R_0$. Naturally, the maximum modulus of compressive stresses on the surface of an inclusion act in its equatorial region, and they are greater the larger the relative size of the inclusion. At the poles of the inclusion, on the contrary, the greatest modulus of stress is observed at the smallest relative size of the inclusion.

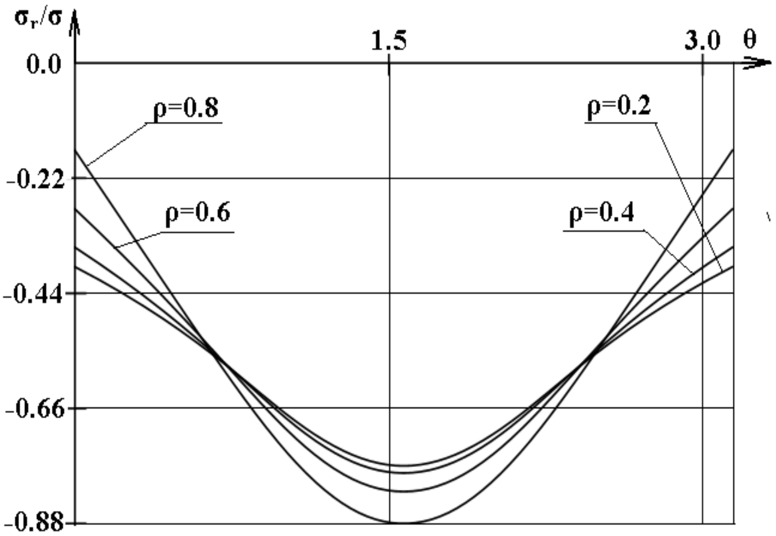

Fig. 1 – Stress graphs on the surface of the brass inclusion. The first type of load.

Fig. 2 shows the distribution of normal stresses on the surface of an aluminum inclusion. The external load is the same as in the previous case. The nature of the stresses remains unchanged, and the absolute values of the stresses change by a small amount. A fundamentally different situation is observed when the materials of the inclusion and the outer sphere are interchanged. Fig. 3 shows the graphs of normal stresses on the surface of the inclusion when the materials of the sphere and the inclusion are aluminum and steel, respectively. Now the maximum modulus of compressive stress is observed at the smallest ratio of the radii $\rho = R_1 / R_0$ of the inclusion and the sphere.

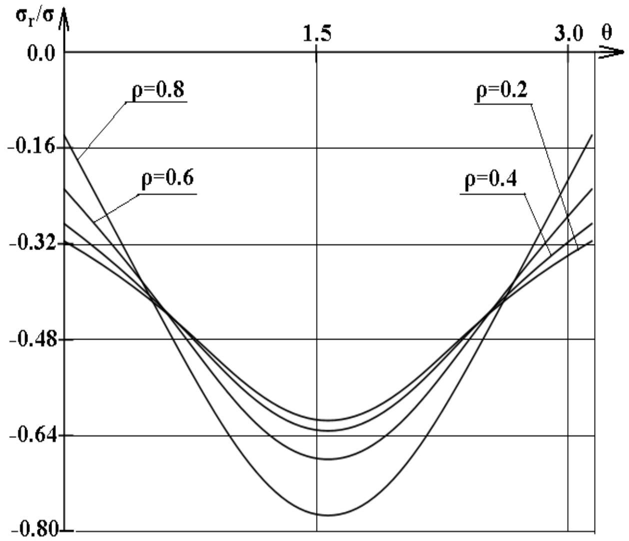

Fig. 2 – Stress graphs σ_r / σ on the surface of an aluminum inclusion. The first type of load.

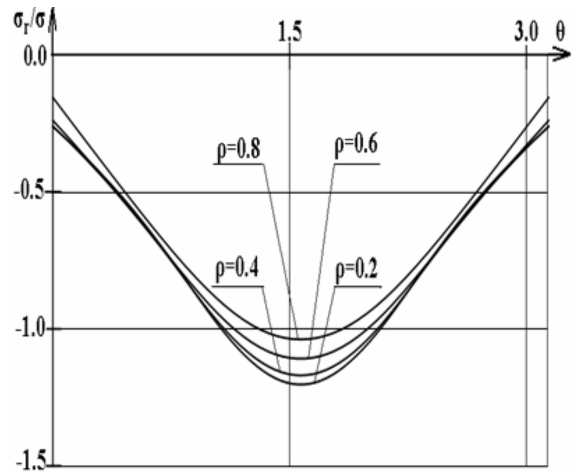

Fig. 3 – Stress graphs σ_r / σ on the surface of a steel inclusion. The first type of load.

The second type of load. Fig. 4, 5 shows the distribution of tangential and normal stresses on the surface of the inclusion (inclusion material is brass) under an external load on the surface of the sphere, which is described by the vector function $\vec{f}(\theta) = \sigma[3\sin\theta\cos\theta\vec{e}_r + \sin^2\theta\vec{e}_\theta]$. Unlike the first type of load, each component of $\vec{f}(\theta)$ is not balanced, but the entire vector satisfies the static condition (37). The tangential stresses and the moduli of normal stresses increase with increasing relative size of the inclusion. The nature of the normal stresses differs from the previous case and is determined by the peculiarity of the external load. With a large relative size of the inclusion, a region is observed on its surface in the vicinity of the poles where the sign of the normal stress differs from the sign of the load.

The practical convergence rate of the method is shown in Tables 1, 2 using the example of calculating normal and tangential stresses on the surface of the inclusion at $\rho = 0.8$ (the second type of load). Here $n_{\max}$ (convergence parameter) is the upper limit of summation when replacing infinite sums with finite ones. The lowest accuracy is observed when calculating $\sigma_r(0)/\sigma$ at $n_{\max} = 30$. It is equal to 0.05 %.

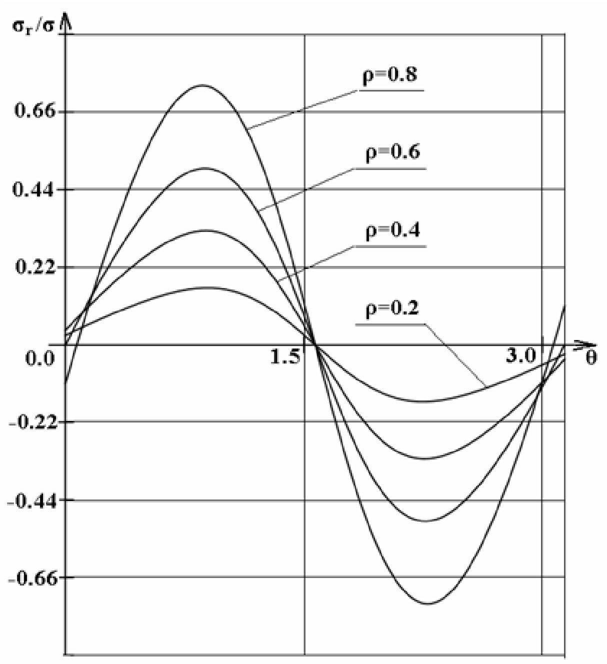

Fig. 4 – Graphs of stresses σ_r/σ on the surface of the inclusion. The second type of load.

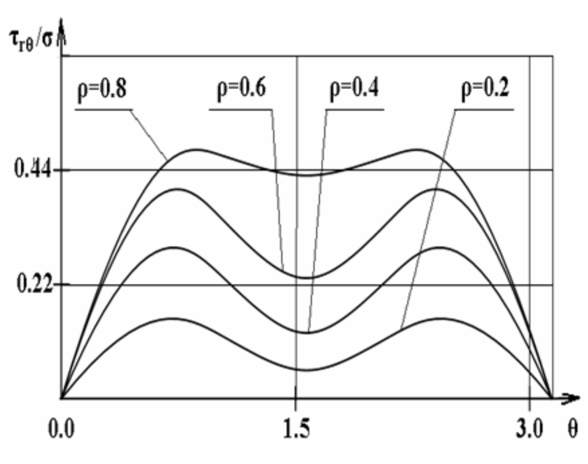

Fig. 5 – Graphs of stresses $\tau_{r\theta}/\sigma$ on the surface of the inclusion. The second type of load.

Table 1 – Practical convergence of the method when calculating $\sigma_r(\theta)/\sigma$, $\rho = 0.8$

$n_{\max} \backslash \theta$	0	$\pi/6$	$\pi/3$
30	−0.10743	0.69313	0.87618
40	−0.10752	0.69314	0.87618
50	−0.10753	0.69314	0.87618

Table 2 – Practical convergence of the method when calculating $\tau_{r\theta}(\theta)/\sigma$, $\rho = 0.8$

$n_{\max} \backslash \theta$	$\pi/6$	$\pi/3$	$\pi/2$
30	0.50210	0.45777	0.40011
40	0.50210	0.45777	0.40011
50	0.50210	0.45777	0.40011

Prospects for further research. The results obtained in the work can be applied to solving a number of problems important for practice: on elastic space with a spherical layer, on a layered piecewise homogeneous sphere, on a sphere with an inclusion under the action of concentrated forces. A separate direction of research is associated with non-concentric spheres and spherical heterogeneity.

Conclusions. For the first time, an exact analytically justified solution of a classical problem of the theory of elasticity – the second axisymmetric boundary value problem in the general formulation for a sphere with a concentric spherical inclusion – was obtained using the Fourier method. The justification for the solution of such a problem and the establishment of its solvability class by the usual Fourier method is based on the analysis of a solvable algebraic system of the sixth order with coefficients that depend on five independent continuous parameters and one discrete one. The general solution of the problem is given in the form of series in terms of axisymmetric vector basis solutions of the Lamé

equation for a sphere, constructed by the authors in one of the previous articles. After transitioning to stresses and satisfying the boundary conditions, a solution system of the above form is obtained. When analyzing the system, a lower estimate for the modulus of its determinant was found for the first time, from which not only the condition for the unique solvability of the system follows, but also estimates of the solutions of the system itself. When estimating the determinant, a new classical inequality was proved for one continuous and one discrete parameter. The next step was to prove a theorem about the conditions that must be imposed on the vector of the external load applied to the surface of the sphere, which ensure the existence of a solution to the problem in a certain class of functions. In the numerical implementation of the solution to the problem, two types of loads on the outer surface of the sphere were considered, which satisfy the equilibrium condition. In the numerical implementation of the solution to the problem, two types of loads on the outer surface of the sphere were considered, which satisfy the equilibrium condition. A computer experiment was conducted with three materials of the ball and the inclusion: steel, brass, aluminum. Graphs of normal and tangential stresses on the surface of the inclusion were obtained, their parametric analysis was performed depending on the geometric and mechanical parameters. The practical convergence of the method was investigated.

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