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STRICT JUSTIFICATION OF THE FOURIER METHOD IN BOUNDARY-VALUE PROBLEMS OF THE THEORY OF ELASTICITY FOR A SYMMETRICALLY LOADED TRANSVERSALLY ISOTROPIC OBLATE SPHEROID AND ITS APPLICATION TO A HOLLOW SPHEROID

For the first time, an exact, well-founded solution by the Fourier method of the second axisymmetric boundary value problems of the theory of elasticity in the general formulation for a transversely isotropic oblate spheroid and a space with a spheroidal cavity has been obtained. The fundamental problem of the justification was the problem of estimating from below the modules of the determinants of the resolving systems for the internal and external problems. The indicated estimates were obtained in this work. The complexity of estimating determinants is due to the fact that they depend on nine parameters that are functionally related to each other, and in addition, two of them are included in the arguments of Legendre functions of the first and second kind. The estimates made it possible to formulate and prove theorems about the solvability conditions of the considered boundary value problems in certain classes of functions. The obtained results are applied to the solution of the second boundary value problem for a transversely isotropic oblate spheroid with a spheroidal cavity, the centers and directions of the axes of which coincide. An arbitrary symmetric balanced load is given on the surfaces of the spheroid, which satisfies a certain condition for the convergence of the series of limit functions developed in terms of Legendre functions. A feature of this problem for a transversely isotropic body is the impossibility of describing spheroidal surfaces with any geometry by a single pair of spheroidal coordinate systems. This means that such a problem can only be solved using the generalized Fourier method. Its application made it possible to reduce the original problem to an infinite system of linear algebraic equations. Thanks to the obtained estimates of the determinants of simply connected problems, the Fredholm property of the system operator in a certain Hilbert space has been proven. The numerical results in the considered problem are obtained in the case of a oblate spheroid with a circular crack. It is assumed that the surface of the spheroid is free of forces, and a constant normal load is applied to the crack. Graphs of normal stresses in the crack plane outside its boundary, as well as the values of stress intensity factor at its boundary, are presented. A parametric analysis of stresses and SIF depending on the geometric parameters of the problem was performed. The practical convergence of the reduction method when solving an infinite system was investigated.

Key words: Fourier method, boundary value problem, transversely isotropic body, oblate spheroid, empty spheroid, justified solution, determinant estimation, Legendre functions, Fredholm operator, circular crack, stress intensity factor, reduction method.

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СТРОГЕ ОБҐРУНТУВАННЯ МЕТОДУ ФУР'Є В КРАЙОВИХ ЗАДАЧАХ ТЕОРІЇ ПРУЖНОСТІ ДЛЯ СИМЕТРИЧНО НАВАНТАЖЕНОГО ТРАНСВЕРСАЛЬНО-ІЗОТРОПНОГО СТИСНУТОГО СФЕРОЇДА ТА ЙОГО ЗАСТОСУВАННЯ ДЛЯ ПОРОЖНЬОГО СФЕРОЇДА

Вперше отримано точний обґрунтований розв'язок методом Фур'є осесиметричних(осе) крайових задач теорії пружності в загальній постановці для трансверсально-ізотропного стиснутого сфероїда і простору зі сфероїдальною порожниною. Принциповою проблемою обґрунтування стала проблема оцінки знизу модулів визначників розв'язуваних систем для внутрішньої та зовнішньої задач. У цій роботі отримано вказані оцінки. Складність оцінювання визначників пов'язана з тим, що вони залежать від дев'яти параметрів, які функціонально пов'язані між собою, до того ж, два з них входять в аргументи функцій Лежандра першого і другого роду. Оцінки дали змогу сформулювати і довести теореми про умови розв'язності розглянутих крайових задач у певних класах функцій. Отримані результати застосовано до розв'язання другої крайової задачі для трансверсально-ізотропного стиснутого сфероїда зі сфероїдальною порожниною, центри і напрями осей яких збігаються. На поверхнях сфероїда задано довільне симетричне врівноважене навантаження, яке задовольняє певну умову збіжності рядів граничних функцій, розвинених за функціями Лежандра. Особливістю цієї задачі для трансверсально-ізотропного тіла є неможливість при будь-якій геометрії сфероїдальних поверхонь описати їх однією парою сфероїдальних систем координат. Це означає, що таку задачу можна розв'язати тільки узагальненим методом Фур'є. Його застосування дало змогу звести вихідну задачу до нескінченної системи лінійних алгебраїчних рівнянь. Завдяки отриманим оцінкам визначників однозв'язних задач доведено фредгольмовість оператора системи в певному гільбертовому просторі. Чисельні результати у розглянутій задачі отримано у випадку стиснутого сфероїда з круговою тріщиною. Вважається, що поверхня сфероїда вільна від зусиль, а до тріщини прикладено стале нормальне навантаження. Наведено графіки нормальних напружень в площині тріщини поза її межею, а також величини коефіцієнтів інтенсивності напружень на її межі. Проведено параметричний аналіз напружень і КІН в залежності від геометричних параметрів задачі. Досліджено практичну збіжність методу редукції при розв'язанні нескінченної системи.

Ключові слова: метод Фур'є, крайова задача, трансверсально-ізотропне тіло, стиснутий сфероїд, порожній сфероїд, обґрунтований розв'язок, оцінка визначника, функції Лежандра, оператор Фредгольма, кругова тріщина, коефіцієнт інтенсивності напружень, метод редукції.

Introduction. In the modern world, more and more attention is paid to the creation of new materials with specified physical and mechanical properties, which has led to the widespread use of composites and anisotropic materials in various fields of technology – from aviation and space to biomedical. Transversely isotropic materials, whose properties are different in a certain direction and in a plane perpendicular to it, have become particularly popular. Such materials effectively simulate real engineering systems, in particular composite and reinforced structures. At the same time, cracks, inclusions, and other inhomogeneities occur in such materials for technological or operational reasons, which significantly complicates the analysis of their strength and reliability. An urgent need for high-precision mathematical modeling also arises when creating nanostructures. Here, local approximate models do not provide a complete and accurate picture of the strength characteristics of the relevant materials. We emphasize that mathematical models of the stress state of complex bodies are usually based on known solutions of boundary value problems of the theory of elasticity for classical simply connected bodies. At the same time, the mathematical justification of the constructed solutions is of fundamental importance, since this is a key element not only of the theoretical rigor, but also of the practical certaintence of the obtained models. Modern computing technologies allow the implementation of numerically complex analytical models however, only reliable mathematical foundations can ensure the correctness of modeling results.

Review of previous research results. The first solutions to problems of the theory of elasticity for bodies with spheroidal surfaces appeared in the first half of the 20th century in the works of the classics. *H. Neuber* published a monograph [1] in 1937, in which he considered bodies with notches. In it, he formulated a number of concepts and notions for the approximate calculation of stresses in the vicinity of notches on the boundary of an elastic body. The concept of the stress concentration coefficient was introduced as the ratio of the nominal stress to the maximum in the vicinity of the notch, and the law of the stress gradient was postulated, according to which a sharp increase in the stress around the notch leads to their significant decrease at the edges of the highly loaded zone. This made it possible to obtain approximate formulas for stresses in the vicinity of notches of various shapes in flat and spatial bodies, in particular for spheroidal notches. Ten years later, *M. A. Sadowsky* and *E. Sternberg* [2] developed an approach using curvilinear orthogonal coordinates and obtained an exact analytical solution for the stresses around a triaxial ellipsoidal cavity in an infinite elastic body under simple loads at infinity. The paper presents a closed-form solution in terms of elliptic Jacobi functions for an arbitrary triaxial (not only spheroidal) cavity. Then *R. H. Edwards* [3] in 1951 considered the problem of an elastic body containing a spheroidal inclusion under a uniform axial load. He showed that for a spherical inclusion the stresses and strains inside it remain uniform even if the elastic modulus and Poisson's ratio of the inclusion differ from those of the matrix. This unexpected result was later generalized by *J. D. Eshelby*. In his classic work [4] Eshelby developed the method of equivalent eigenstrains and showed that for an ellipsoidal inclusion a constant eigenstrain gives rise to a uniform field of stresses and strains inside the inclusion. *A. I. Lur'e* in his monograph [5] proposed a method for solving the problem for an elastic space with an ellipsoidal cavity based on *Papkovich – Neiber potentials*. However, a later analysis by *Xu*, *Zhao*, and *Wang* [6] showed that the harmonic functions chosen by Lur'e lead to an unsolvable algebraic system. These authors used the Eshelby equivalent inclusion method to construct a correct solution to the problem with an ellipsoidal cavity and wrote out new Papkovich – Neiber harmonic functions that satisfy the boundary conditions on the surface of the cavity. *Yu. M. Podilchuk* in his monograph [7] constructed exact analytical solutions of the Lamé equation for the interior and exterior of prolate and oblate spheroids by the Fourier method by choosing harmonic functions in the Papkovich – Neiber representation. In one of Podilchuk's works, the stressed state of a medium with an absolutely rigid ellipsoidal inclusion (the limiting case of given zero deformations in the inclusion) was analyzed. His other work [8] (jointly with *V. S. Kyrylyuk*) investigated the state of an elastic body with an ellipsoidal cavity according to the polynomial law of loading at infinity. Later, the same ideas were used in constructing exact solutions for transversely isotropic spheroids [9]. The construction of exact analytical basic solutions of the *Lamé equation* for spheroids was first made in [10]. For transversely isotropic spheroids, similar solutions were constructed in [11]. The problem of substantiating exact solutions for isotropic and transversely isotropic spheroids was first posed and solved in [12, 13]. In [14], a general three-dimensional analytical solution of the problem of a transversely isotropic space containing a spheroidal cavity with given asymmetric displacements on the surface of the cavity and vanishing stresses and displacements at infinity is presented. The approach is applied to obtain a solution to the problem of a transversely isotropic space containing a perfectly rigid spheroidal inclusion, where the space at infinity is subjected to uniform tension in the direction perpendicular to the axis of elastic symmetry of the material. In the article [15], the problem of stress concentration around a triaxial ellipsoidal cavity in transversely isotropic materials was solved using the Eshelby equivalent inclusion method. The authors of the work [16] *H. Amstutz* and *M. Vormwald* demonstrate a modern engineering approach to the analysis of the stress state in the vicinity of an inclusion in isotropic space. An elastic inclusion in the form of prolate spheroid under axial tension was considered and an analytical solution was constructed, based on the solution for the cavity, in which the displacement in the vicinity of the cavity was obtained by direct integration of the formulas for the connection between deformations and stresses. In the article [17], the stress concentration problem for an elastic transversely isotropic medium containing an arbitrarily oriented spheroidal inclusion (inhomogeneity) is solved. The stress state in elastic space is represented as a superposition of the main state and the perturbed state caused by the inhomogeneity. The problem is solved using the method of equivalent inclusions, the triple Fourier transform in spatial variables, and the Green's function for an infinite anisotropic medium.

We emphasize that all the listed works do not consider the problem of substantiating solutions to the second boundary value problem in the general symmetric formulation for a transversely isotropic compressed spheroid and a space with a spheroidal cavity.

Statement of the external boundary value problem. Consider a three-dimensional space in which an arbitrary point O and a cylindrical coordinate system (ρ, φ, z) with the origin at this point are fixed. Next, we will consider two transversely isotropic elastic bodies, the anisotropy axes of which coincide with the axis Oz. The regions occupied by the bodies are denoted by

$$\Omega^{\pm} = \left\{ (\rho, \varphi, z) : \frac{\rho^2}{d_2^2} + \frac{z^2}{d_1^2} \gtrless 1 \right\},$$

where $d_1 / d_2 < \min\{\sqrt{\nu_1}, \sqrt{\nu_2}\}$, ν_1, ν_2 are the positive roots of the equation

$$c_{11}c_{44}\nu^2 - (c_{11}c_{33} - 2c_{13}c_{44} - c_{13}^2)\nu + c_{33}c_{44} = 0 .$$

Here c_{ij} are the elastic constants of the transversely isotropic material of the bodies under consideration (they are all considered positive). The system of equilibrium equations in the displacements of a transversely isotropic body in the case of axial symmetry has the form

$$\left[c_{11}\left(\Delta_2-\frac{1}{\rho^2}\right)+c_{44}\frac{\partial^2}{\partial z^2}\right]V_\rho+(c_{13}+c_{44})\frac{\partial^2 V_z}{\partial\rho\partial z}=0\,, \quad (1)$$

$$\left[c_{44}\Delta_2+c_{33}\frac{\partial^2}{\partial z^2}\right]V_z+(c_{13}+c_{44})\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V_\rho}{\partial z}\right)=0\,. \quad (2)$$

Above (V_ρ, V_z) – are the components of the axisymmetric displacement vector in cylindrical coordinates, $\Delta_2\equiv\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}$ – an axisymmetric variant of the two-dimensional Laplace operator in polar coordinates.

To describe the domains $\Omega^\pm$, it is convenient to use two oblate spheroidal coordinate systems $\{(\tilde\xi_s,\tilde\eta_s,\varphi)\}_{s=1}^2$, the coordinates of which are related to the cylindrical coordinates by the formulas

$$\rho=c_s\mathrm{ch}\tilde\xi_s\sin\tilde\eta_s\,,\quad \frac{z}{\sqrt{\nu_s}}=c_s\mathrm{sh}\tilde\xi_s\cos\tilde\eta_s\,.$$

Here $c_s>0$ is the parameters of the spheroidal systems, $\tilde\xi_s\in[0,\infty)$, $\tilde\eta_s\in[0,\pi]$, $\varphi\in[0,2\pi]$. Then the surface $\partial\Omega^\pm$ is given by the equation $\tilde\xi_s=\tilde\xi_s^0$, where $c_s\mathrm{ch}\tilde\xi_s^0=d_2$, $\sqrt{\nu_s}c_s\mathrm{sh}\tilde\xi_s^0=d_1$ and on this surface $\tilde\eta_s=\tilde\eta$.

Let us consider the second axisymmetric boundary value problem of the theory of elasticity for the system of equations (1), (2) in the domain Ω^+, when the load is given on the boundary of the domain $\partial\Omega^+$

$$F\vec V\Big|_{\tilde\xi_s=\tilde\xi_s^0}=\frac{c_{44}}{H}\sum_{n=0}^{\infty}\left[b_n^{(1)}P_n^{(1)}\left(\cos\tilde\eta\right)\mathbf{e}_\rho+b_nP_n\left(\cos\tilde\eta\right)\mathbf{e}_z\right], \quad (3)$$

where $H=(d_1^2\sin^2\tilde\eta+d_2^2\cos^2\tilde\eta)^{1/2}$, and at infinity the regularity conditions are satisfied. The conditions for the coefficients of the series (3) will be given later.

Construction of a solution to the external boundary value problem. In the article [11], general basis sets of transversely isotropic displacements of canonical bodies in all curvilinear coordinate systems in which the variables in the Laplace equation are separated are constructed. For oblate spheroidal coordinates, axisymmetric versions of such sets have the form

$$\vec V_{s,n}^{\pm(6)}(\tilde\xi_{js},\tilde\eta_{js})=\frac{-ic_j}{2n+1}\vec\nabla_s[u_{n-1}^{\pm(6)}(\tilde\xi_{js},\tilde\eta_{js})-u_{n+1}^{\pm(6)}(\tilde\xi_{js},\tilde\eta_{js})],\quad n=0,1,...,\ \ s=1,2\,, \quad (4)$$

where

$$u_n^{\pm(6)}(\xi,\eta)=\begin{Bmatrix}Q_n(i\mathrm{sh}\tilde\xi)\\ P_n(i\mathrm{sh}\tilde\xi)\end{Bmatrix}P_n(\cos\tilde\eta)\,;\ \vec\nabla_s=\vec e_\rho\frac{\partial}{\partial\rho}+k_s\vec e_z\frac{\partial}{\partial z}\,,\ k_s=\frac{c_{11}\nu_s-c_{44}}{c_{13}+c_{44}}\,,\ s=1,2\,,$$

$P_n(x), Q_n(x)$ – Legendre functions of the first and second kind; $\{\vec e_\rho,\vec e_z\}$ – unit base vectors of the cylindrical coordinate system. The displacements $\vec V_{s,n}^{\pm(6)}$ are solutions of the system (1), (2), regular in the domains $\Omega^\pm$. The displacements (4) in coordinates have the following form:

$$\vec V_{s,n}^{\pm(6)}(\tilde\xi_{js},\tilde\eta_{js})=u_n^{\pm(6)1}(\tilde\xi_{js},\tilde\eta_{js})\vec e_\rho-\frac{k_s}{\sqrt{\nu_s}}u_n^{\pm(6)}(\tilde\xi_{js},\tilde\eta_{js})\vec e_z\,,\ \ s=1,2\,, \quad (5)$$

where

$$u_n^{\pm(6)1}(\tilde\xi,\tilde\eta)=\begin{Bmatrix}Q_n^1(i\mathrm{sh}\tilde\xi)\\ P_n^1(i\mathrm{sh}\tilde\xi)\end{Bmatrix}P_n^{-1}(\cos\tilde\eta)\,.$$

The stresses on the surface $\partial\Omega^\pm$ with the normal $\vec n_{\tilde\xi_s}$, which correspond to the displacement vectors $\vec V_{s,n}^{\pm(6)}(\tilde\xi_{js},\tilde\eta_{js})$, are calculated by the formula

$$F\vec V_{s,n}^{\pm(6)}(\tilde\xi_s^{(0)},\eta_s)=\frac{\sqrt{\nu_s}}{H}\left\{c_{44}\frac{k_s+1}{\nu_s}\left[\frac{\partial}{\partial\tilde\xi_s}u_n^{\pm(6)1}(\tilde\xi_s,\tilde\eta_s)\vec e_\rho-\sqrt{\nu_s}\frac{\partial}{\partial\tilde\xi_s}u_n^{\pm(6)}(\tilde\xi_s,\tilde\eta_s)\vec e_z\right]+\right.$$
$$\left.+(c_{12}-c_{13}\frac{k_s}{\nu_s})\mathrm{th}\tilde\xi_s^{(0)}u_n^{\pm(6)1}(\tilde\xi_s,\tilde\eta_s)\vec e_\rho\right\}. \quad (6)$$

We will look for the solution to the problem in the form of a series with external basis displacements (4)

$$\vec{V}(\rho, z) = \sum_{s=1}^{2}\sum_{n=0}^{\infty} a_{s,n}\vec{V}_{s,n}^{+(6)}(\tilde{\xi}_s, \tilde{\eta}_s), \tag{7}$$

where $\{a_{s,n}\}_{s=1,n=0}^{2,\infty}$ are unknown coefficients. Let us proceed in formula (7) to the surface stresses according to formula (6), after which we satisfy the boundary condition (3). As a result, we obtain an algebraic system with respect to $\{a_{s,n}\}_{s=1,n=0}^{2,\infty}$. After some transformations with the coefficients of the obtained system, we will have

$$\sum_{s=1}^{2} a_{s,0}Q_0^{(1)}(i\overline{q}_s) = 0, \ \sum_{s=1}^{2} a_{s,0}\frac{c_{11}\nu_s + c_{13}}{c_{13}+c_{44}}Q_0^{(1)}(i\overline{q}_s) = b_0; \tag{8}$$

$$\sum_{s=1}^{2} a_{s,n}\left[\frac{k_s+1}{\sqrt{\nu_s}}n(n+1)Q_n(i\overline{q}_s) - \frac{c_{11}-c_{12}}{c_{44}}\frac{d_1}{d_2}Q_n^{(1)}(i\overline{q}_s)\right] = n(n+1)b_n^{(1)}, \ n \ge 1; \tag{9}$$

$$\sum_{s=1}^{2} a_{s,n}(k_s+1)Q_n^{(1)}(i\overline{q}_s) = b_n, \ n \ge 1. \tag{10}$$

Here and below marked $\operatorname{ch}\tilde{\xi}_s^0 = q_s$, $\operatorname{sh}\tilde{\xi}_s^0 = \overline{q}_s$.

Analysis of the resolving system for the exterior problem. The following theorem provides an analysis of the solvability of system (8) – (10) and the conditions for the existence of a classical solution to the boundary value problem (1) – (3).

Theorem 1. *For $\nu_1 \ne \nu_2$ system (8) – (10) is uniquely solvable. For its determinant the estimate*

$$\left|\Delta_n^{+(2)6}(\tilde{\xi}_1^0, \tilde{\xi}_2^0)\right| \ge \frac{c_{11}-c_{12}}{c_{44}}\frac{c_{11}|\nu_1-\nu_2|}{c_{13}+c_{44}}\frac{d_1}{d_2}\left|Q_n^{(1)}(i\overline{q}_1)Q_n^{(1)}(i\overline{q}_2)\right|, \ n \ge 1 \tag{11}$$

holds. When the condition

$$\sum_{n=0}^{\infty} n(n|b_n^{(1)}| + |b_n|) < \infty \tag{12}$$

is met, there is a solution to problem (1) – (3) in the domain Ω^+ by the Fourier method in the form (7), *which belongs to the space $C^2(\Omega^+)\bigcap C^1(\overline{\Omega^+})$.*

Proof of the theorem.

First of all, we note that the first equation (8) guarantees the regularity of the vector function (7). System (8) has a solution

$$a_{1,0} = ib_0\frac{d_2}{c_1}\frac{c_{13}+c_{44}}{c_{11}(\nu_2-\nu_1)}, \ a_{2,0} = ib_0\frac{d_2}{c_2}\frac{c_{13}+c_{44}}{c_{11}(\nu_1-\nu_2)}. \tag{13}$$

To analyze the solvability of system (9), (10), we consider the determinant of this system

$$\Delta_n^{+(2)6}(\tilde{\xi}_1^0, \tilde{\xi}_2^0) = \begin{vmatrix} \frac{k_1+1}{\sqrt{\nu_1}}n(n+1)Q_n(i\overline{q}_1) - \frac{c_{11}-c_{12}}{c_{44}}\frac{d_1}{d_2}Q_n^{(1)}(i\overline{q}_1) & (k_1+1)Q_n^{(1)}(i\overline{q}_1) \\ \frac{k_2+1}{\sqrt{\nu_2}}n(n+1)Q_n(i\overline{q}_2) - \frac{c_{11}-c_{12}}{c_{44}}\frac{d_1}{d_2}Q_n^{(1)}(i\overline{q}_2) & (k_2+1)Q_n^{(1)}(i\overline{q}_2) \end{vmatrix} =$$

$$= (k_1+1)(k_2+1)Q_n^{(1)}(i\overline{q}_1)Q_n^{(1)}(i\overline{q}_2)\left\{n(n+1)\left[\frac{1}{\sqrt{\nu_1}}\frac{Q_n(i\overline{q}_1)}{Q_n^{(1)}(i\overline{q}_1)} - \frac{1}{\sqrt{\nu_2}}\frac{Q_n(i\overline{q}_2)}{Q_n^{(1)}(i\overline{q}_2)}\right] + \frac{c_{11}-c_{12}}{c_{44}}\frac{d_1}{d_2}\left(\frac{1}{k_2+1} - \frac{1}{k_1+1}\right)\right\}. \tag{14}$$

Since, as was proved in [18], the function

$$\sigma(\nu) = \frac{1}{\sqrt{\nu}}\frac{Q_n(i\overline{q})}{Q_n^{(1)}(i\overline{q})}$$

is monotonically increasing, then both terms in curly brackets (14) have the same sign, therefore the determinant is non-zero and the estimate (11) is satisfied. Then the system (9), (10) is uniquely solvable and its solutions have the form

$$a_{1,n} = -[\Delta_n^{+(2)6}(\tilde{\xi}_1^0, \tilde{\xi}_2^0)]^{-1}\left\{b_n\left[\frac{k_2+1}{\sqrt{\nu_2}}n(n+1)Q_n(i\overline{q}_2) - \frac{c_{11}-c_{12}}{c_{44}}\frac{d_1}{d_2}Q_n^{(1)}(i\overline{q}_2)\right] - \right.$$

$$\left. - b_n^{(1)}n(n+1)(k_2+1)Q_n^{(1)}(i\overline{q}_2)\right\}, \ n \ge 1; \tag{15}$$

$$a_{2,n}=-[\Delta_n^{+(2)6}(\tilde{\xi}_1^0,\tilde{\xi}_2^0)]^{-1}\left\{-b_n\left[\frac{k_1+1}{\sqrt{\nu_1}}n(n+1)Q_n(i\overline{q}_1)-\frac{c_{11}-c_{12}}{c_{44}}\frac{d_1}{d_2}Q_n^{(1)}(i\overline{q}_1)\right]+\right.$$

$$\left.+b_n^{(1)}n(n+1)(k_1+1)Q_n^{(1)}(i\overline{q}_1)\right\},\ n\geq 1; \qquad (16)$$

Since for Legendre functions of the second kind the estimates are

$$\frac{d_1}{\sqrt{\nu}(n+1)d_2}=\frac{\overline{q}}{(n+1)q}<\frac{|Q_n(i\overline{q})|}{|Q_n^{(1)}(i\overline{q})|}<\frac{q}{(n+1)\overline{q}}=\frac{\sqrt{\nu}d_2}{(n+1)d_1},\ \overline{q}>0,\ n\in\mathbb{N}, \qquad (17)$$

then the following estimates follow from the obtained solutions and formulas (11), (17):

$$|a_{1,n}|\leq\frac{c_{13}+c_{44}}{c_{11}}\frac{c_{44}}{c_{11}-c_{12}}\frac{1}{|\nu_1-\nu_2|}\frac{d_2}{d_1|Q_n^{(1)}(i\overline{q}_1)|}\left\{|b_n|\left[\frac{k_2+1}{\sqrt{\nu_2}}n(n+1)\frac{|Q_n(i\overline{q}_2)|}{|Q_n^{(1)}(i\overline{q}_2)|}+\right.\right.$$

$$\left.\left.+\frac{c_{11}-c_{12}}{c_{44}}\frac{d_1}{d_2}\right]+|b_n^{(1)}|n(n+1)(k_2+1)\right\}<\frac{C_1n(|b_n^{(1)}|(n+1)+|b_n|)}{|Q_n^{(1)}(i\overline{q}_1)|}, \qquad (18)$$

$$|a_{2,n}|\leq\frac{c_{13}+c_{44}}{c_{11}}\frac{c_{44}}{c_{11}-c_{12}}\frac{1}{|\nu_1-\nu_2|}\frac{d_2}{d_1|Q_n^{(1)}(i\overline{q}_2)|}\left\{|b_n|\left[\frac{k_1+1}{\sqrt{\nu_1}}n(n+1)\frac{|Q_n(i\overline{q}_1)|}{|Q_n^{(1)}(i\overline{q}_1)|}+\right.\right.$$

$$\left.\left.+\frac{c_{11}-c_{12}}{c_{44}}\frac{d_1}{d_2}\right]+|b_n^{(1)}|n(n+1)(k_1+1)\right\}<\frac{C_2n(|b_n^{(1)}|(n+1)+|b_n|)}{|Q_n^{(1)}(i\overline{q}_1)|}, \qquad (19)$$

where the constants C_s are positive and depend only on s.

Now we can estimate the terms of the series (7) using estimates (17) – (19) and the uniform estimate

$$\frac{|Q_n^{(1)}(i\text{sh}\tilde{\xi})|}{|Q_n^{(1)}(i\overline{q})|}\leq\frac{q^{n+1}}{(\text{ch}\tilde{\xi})^{n+1}},\ \tilde{\xi}\geq\tilde{\xi}^{(0)},\ n\in\mathbb{N}, \qquad (20)$$

that follows from the integral representation of the Legendre function of the second kind. Therefore, we have

$$\left|\sum_{s=1}^{2}a_{s,n}\vec{V}_{s,n}^{+(6)}(\tilde{\xi}_s,\tilde{\eta}_s)\right|\leq\sum_{s=1}^{2}|a_{s,n}||\vec{V}_{s,n}^{+(6)}(\tilde{\xi}_s,\tilde{\eta}_s)|\leq$$

$$\leq\sum_{s=1}^{2}|a_{s,n}|\left[\frac{|Q_n^{(1)}(i\mathbf{sh}\tilde{\xi}_s)||P_n^{(1)}(\cos\tilde{\eta}_s)|}{n(n+1)}+|Q_n(i\mathbf{sh}\tilde{\xi}_s)||P_n(\cos\tilde{\eta}_s)|\right]\leq\sum_{s=1}^{2}\tilde{C}_s(n|b_n^{(1)}|+|b_n|)\left(\frac{q_s}{\text{ch}\tilde{\xi}_s}\right)^{n+1}, \qquad (21)$$

where the constants $\tilde{C}_s$ are positive and depend only on s.

Estimate (21) shows that when condition (12) is satisfied, series (7) converges absolutely and uniformly at $\tilde{\xi}_s\geq\tilde{\xi}_s^{(0)}$ and can be infinitely differentiated by terms in the domain $\tilde{\xi}_s>\tilde{\xi}_s^{(0)}$. The latter means that formula (7) specifies a solution to the boundary value problem (1) – (3) in the domain Ω^+, which belongs to the class of functions $C^2(\Omega^+)\bigcap C^1(\overline{\Omega^+})$.

The statement of the internal boundary value problem and construction of its solution. Now let us consider the second boundary value problem for the system of equations (1), (2) in a transversely isotropic oblate spheroid

$$\Omega^-=\left\{(x,y,z):\frac{\rho^2}{d_2^2}+\frac{z^2}{d_1^2}<1\right\}$$

with boundary condition (3). We assume that the conditions formulated in the statement of the external problem are met, and the load (3) is balanced. To describe the domain Ω^-, we will use the two systems of oblate spheroidal coordinates $\{(\tilde{\xi}_s,\tilde{\eta}_s,\varphi)\}_{s=1}^2$ introduced above and the same notations.

We will look for the solution to the problem in the form of a series

$$\vec{V}(\rho,z)=a_{1,0}\vec{V}_{1,0}^{-(6)}(\tilde{\xi}_1,\tilde{\eta}_1)+\sum_{s=1}^{2}\sum_{n=1}^{\infty}a_{s,n}\vec{V}_{s,n}^{-(6)}(\tilde{\xi}_s,\tilde{\eta}_s), \qquad (22)$$

where $\{a_{1,0},a_{s,n}\}_{s=1,n=1}^{2,\infty}$ are the unknown coefficients. Let us proceed in formula (22) to the surface stresses according to formula (6), after which we satisfy the boundary conditions (3). After transforming the coefficients in the Legendre functions, we obtain an algebraic system with respect to $\{a_{s,n}\}_{s=1,n=1}^{2,\infty}$

$$\sum_{s=1}^{2} a_{s,n}\left[\frac{k_s+1}{\sqrt{\nu_s}}n(n+1)P_n(i\overline{q}_s)-\frac{c_{11}-c_{12}}{c_{44}}\frac{d_1}{d_2}P_n^{(1)}(i\overline{q}_s)\right]=-n(n+1)b_n^{(1)}\ ,\quad n\geq 1\ ; \tag{23}$$

$$\sum_{s=1}^{2} a_{s,n}(k_s+1)P_n^{(1)}(i\overline{q}_s)=-b_n\ ,\quad n\geq 1\ . \tag{24}$$

The boundary condition at $n=0$ is satisfied automatically for any $a_{1,0}$, since the statics conditions lead to the equality $b_0=0$. In turn, this leads to the standard result for the inner problem – the solution of the problem is determined uniquely up to a rigid displacement.

Analysis of the resolving system for the interior problem. The following theorem provides an analysis of the solvability of system (23), (24) and the conditions for the existence of a classical solution to the boundary value problem.

Theorem 2. *For $\nu_1\neq\nu_2$, $n\geq 1$, the system (23), (24) is uniquely solvable. For its determinant, the estimate*

$$|\,\Delta_n^{-(2)6}(\tilde{\xi}_1^0,\tilde{\xi}_2^0)\,|\geq|\,P_n^{(1)}(i\overline{q}_1)\,||\,P_n^{(1)}(i\overline{q}_2)\,|\,\frac{c_{11}c_{33}-c_{13}^2}{c_{44}(c_{13}+c_{44})}\sigma\frac{d_1}{d_2}\frac{|\,\nu_1-\nu_2\,|}{\max(\nu_1^2,\nu_2^2)} \tag{25}$$

holds. *When the condition*

$$\sum_{n=0}^{\infty} n(n\,|\,b_n^{(1)}\,|+|\,b_n\,|)<\infty \tag{26}$$

is met and the load (3) is balanced, then there is a solution to problem (1) – (3) in the domain Ω^- by the Fourier method in the form (22), *which belongs to the space $C^2(\Omega^-)\bigcap C^1(\overline{\Omega^-})$* .

Proof of the theorem.

To analyze the solvability of system (25), (26), we consider the determinant of this system

$$\Delta_n^{-(2)6}(\tilde{\xi}_1^0,\tilde{\xi}_2^0)=\begin{vmatrix}\dfrac{k_1+1}{\sqrt{\nu_1}}n(n+1)P_n(i\overline{q}_1)-\dfrac{c_{11}-c_{12}}{c_{44}}\dfrac{d_1}{d_2}P_n^{(1)}(i\overline{q}_1) & (k_1+1)P_n^{(1)}(i\overline{q}_1)\\ \dfrac{k_2+1}{\sqrt{\nu_2}}n(n+1)P_n(i\overline{q}_2)-\dfrac{c_{11}-c_{12}}{c_{44}}\dfrac{d_1}{d_2}P_n^{(1)}(i\overline{q}_2) & (k_2+1)P_n^{(1)}(i\overline{q}_2)\end{vmatrix}=$$

$$=P_n^{(1)}(i\overline{q}_1)P_n^{(1)}(i\overline{q}_2)\left\{n(n+1)(k_1+1)(k_2+1)\left[\frac{1}{\sqrt{\nu_1}}\frac{P_n(i\overline{q}_1)}{P_n^{(1)}(i\overline{q}_1)}-\frac{1}{\sqrt{\nu_2}}\frac{P_n(i\overline{q}_2)}{P_n^{(1)}(i\overline{q}_2)}\right]+\frac{c_{11}-c_{12}}{c_{44}}\frac{d_1}{d_2}(k_1-k_2)\right\}. \tag{27}$$

After some transformations, the determinant can be represented in this form

$$\Delta_n^{-(2)6}(\tilde{\xi}_1^0,\tilde{\xi}_2^0)=$$

$$=P_n^{(1)}(i\overline{q}_1)P_n^{(1)}(i\overline{q}_2)\frac{c_{11}c_{33}-c_{13}^2}{c_{44}(c_{13}+c_{44})}\left\{n(n+1)\left[\frac{1}{\sqrt{\nu_1}}\frac{P_n(i\overline{q}_1)}{P_n^{(1)}(i\overline{q}_1)}-\frac{1}{\sqrt{\nu_2}}\frac{P_n(i\overline{q}_2)}{P_n^{(1)}(i\overline{q}_2)}\right]+(1-\sigma)\frac{d_1}{d_2}\left(\frac{1}{\nu_2}-\frac{1}{\nu_1}\right)\right\}, \tag{28}$$

where σ is Poisson's ratio in the isotropy plane. Note that from the relations between the elastic constants it follows that the constant $c_{11}c_{33}-c_{13}^2>0$.

Consider the function

$$\zeta(\nu)=n(n+1)\frac{1}{\sqrt{\nu}}\frac{P_n(i\overline{q})}{P_n^{(1)}(i\overline{q})}-\frac{1}{\sqrt{\nu}}\frac{\overline{q}}{q}. \tag{29}$$

Let's check it for monotonicity. To do this, we find its derivative

$$\frac{d\zeta(\nu)}{d\nu}=-\frac{n(n+1)}{2\sqrt{\nu^3}[P_n^{(1)}(i\overline{q})]^2}\left\{q^2P_n(i\overline{q})P_n^{(1)}(i\overline{q})+\left[\overline{q}q-\frac{2\overline{q}}{qn(n+1)}\right][P_n^{(1)}(i\overline{q})]^2-\overline{q}qn(n+1)[P_n(i\overline{q})]^2\right\}.$$

It can be proved that the expression

$$(-i)^{2n}\left\{q^2P_n(i\overline{q})P_n^{(1)}(i\overline{q})+\left[\overline{q}q-\frac{2\overline{q}}{qn(n+1)}\right][P_n^{(1)}(i\overline{q})]^2-\overline{q}qn(n+1)[P_n(i\overline{q})]^2\right\}$$

when $n\geq 1$ is positive, i.e. $\dfrac{d\zeta(\nu)}{d\nu}<0$, and the function $\zeta(\nu)$ decreases on the semi-axis $\tilde{\xi}\in(0,\infty)$. Because

$$\Delta_n^{-(2)6}(\tilde{\xi}_1^0,\tilde{\xi}_2^0)=P_n^{(1)}(i\overline{q}_1)P_n^{(1)}(i\overline{q}_2)\frac{c_{11}c_{33}-c_{13}^2}{c_{44}(c_{13}+c_{44})}\left\{[\zeta(\nu_1)-\zeta(\nu_2)]+\sigma\frac{d_1}{d_2}\left(\frac{1}{\nu_1}-\frac{1}{\nu_2}\right)\right\} \tag{30}$$

and the function $\dfrac{1}{\nu}$ also decreases, then both terms in the curly brackets (30) have the same sign, so the estimate (25) is correct. Therefore, the determinant $\Delta_n^{-(2)6}(\tilde{\xi}_1^0, \tilde{\xi}_2^0)$ for n>1 is nonzero. Then the system (23), (24) is uniquely solvable and its solutions have the form ($n \ge 1$)

$$a_{1,n} = [\Delta_n^{-(2)6}(\tilde{\xi}_1^0, \tilde{\xi}_2^0)]^{-1} \left\{ b_n \left[\frac{k_2+1}{\sqrt{\nu_2}} n(n+1) P_n(i\overline{q}_2) - \frac{c_{11}-c_{12}}{c_{44}} \frac{d_1}{d_2} P_n^{(1)}(i\overline{q}_2) \right] + b_n^{(1)} n(n+1)(k_2+1) P_n^{(1)}(i\overline{q}_2) \right\}, \quad (31)$$

$$a_{2,n} = [\Delta_n^{-(2)6}(\tilde{\xi}_1^0, \tilde{\xi}_2^0)]^{-1} \left\{ -b_n \left[\frac{k_1+1}{\sqrt{\nu_1}} n(n+1) P_n(i\overline{q}_1) - \frac{c_{11}-c_{12}}{c_{44}} \frac{d_1}{d_2} P_n^{(1)}(i\overline{q}_1) \right] + b_n^{(1)} n(n+1)(k_1+1) P_n^{(1)}(i\overline{q}_1) \right\}. \quad (32)$$

Given the estimate for Legendre functions of the first kind

$$\frac{d_1}{\sqrt{\nu_s} n d_2} = \frac{\overline{q}_s}{n q_s} < \frac{|P_n(i\overline{q}_s)|}{|P_n^{(1)}(i\overline{q}_s)|} < \frac{q_s}{n\overline{q}_s} = \frac{\sqrt{\nu_s} d_2}{n d_1}, \quad (33)$$

from the obtained solutions (31), (32) and formula (25) the following estimates follow:

$$|a_{1,n}| \le \frac{c_{13}+c_{44}}{c_{11}} \frac{c_{44}}{c_{11}-c_{12}} \frac{1}{|\nu_1-\nu_2|} \frac{d_2}{d_1 |P_n^{(1)}(i\overline{q}_1)|} \left\{ |b_n| \left[\frac{k_2+1}{\sqrt{\nu_2}} n(n+1) \frac{|P_n(i\overline{q}_2)|}{|P_n^{(1)}(i\overline{q}_2)|} + \right.\right.$$
$$\left.\left. + \frac{c_{11}-c_{12}}{c_{44}} \frac{d_1}{d_2} \right] + |b_n^{(1)}| n(n+1)(k_2+1) \right\} < \frac{C_1 n(|b_n^{(1)}|(n+1) + |b_n|)}{|P_n^{(1)}(i\overline{q}_1)|}, \quad (34)$$

$$|a_{2,n}| \le \frac{c_{13}+c_{44}}{c_{11}} \frac{c_{44}}{c_{11}-c_{12}} \frac{1}{|\nu_1-\nu_2|} \frac{d_2}{d_1 |P_n^{(1)}(i\overline{q}_2)|} \left\{ |b_n| \left[\frac{k_1+1}{\sqrt{\nu_1}} n(n+1) \frac{|P_n(i\overline{q}_1)|}{|P_n^{(1)}(i\overline{q}_1)|} + \right.\right.$$
$$\left.\left. + \frac{c_{11}-c_{12}}{c_{44}} \frac{d_1}{d_2} \right] + |b_n^{(1)}| n(n+1)(k_1+1) \right\} < \frac{C_2 n(|b_n^{(1)}|(n+1) + |b_n|)}{|P_n^{(1)}(i\overline{q}_1)|}, \quad (35)$$

where the constants C_s are positive and depend only on s.

Now we can estimate the terms of the series (22) using formulas (33) – (35) and the uniform estimate

$$\left| \frac{P_n^{(1)}(i\,\mathrm{sh}\tilde{\xi})}{P_n^{(1)}(i\,\mathrm{sh}\tilde{\xi}^{(0)})} \right| \le \left(\frac{\mathrm{ch}\tilde{\xi}}{\mathrm{ch}\tilde{\xi}^{(0)}} \right)^n, \ \tilde{\xi} \le \tilde{\xi}^{(0)}, \ n \in \mathbb{N}.$$

As a result, we have

$$\left| \sum_{s=1}^{2} a_{s,n} \vec{V}_{s,n}^{-(6)}(\tilde{\xi}_s, \tilde{\eta}_s) \right| \le \sum_{s=1}^{2} |a_{s,n}| |\vec{V}_{s,n}^{-(6)}(\tilde{\xi}_s, \tilde{\eta}_s)| \le$$
$$\le \sum_{s=1}^{2} |a_{s,n}| \left[\frac{|P_n^{(1)}(i\,\mathrm{sh}\tilde{\xi}_s)| |P_n^{(1)}(\cos\tilde{\eta}_s)|}{n(n+1)} + |P_n(i\,\mathrm{sh}\tilde{\xi}_s)| |P_n(\cos\tilde{\eta}_s)| \right] \le \sum_{s=1}^{2} \tilde{C}_s (n|b_n^{(1)}| + |b_n|) \left(\frac{q_s}{\mathrm{ch}\tilde{\xi}_s} \right)^{n+1}, \quad (36)$$

where the constants $\tilde{C}_s$ are positive and depend only on s. Estimate (36) shows that under condition (26) the series (22) converges absolutely and uniformly at $\tilde{\xi}_s \le \tilde{\xi}_s^{(0)}$ and it can be infinitely differentiated by terms in the domain $\tilde{\xi}_s < \tilde{\xi}_s^{(0)}$. The latter means that formula (22) specifies a solution to the boundary value problem (1) – (3) in the domain Ω^-, which belongs to the class of functions $C^2(\Omega^-) \cap C^1(\overline{\Omega^-})$.

Transversely isotropic oblate spheroid with a spheroidal cavity. We use the results obtained above to solve the second boundary value problem in a transversely isotropic oblate spheroid with a spheroidal cavity, which is under the action of an arbitrary balanced axisymmetric load. We denote the domain occupied by the spheroid by Ω, and the surfaces of the cavity and spheroid by Γ_1 and Γ_2. We assume that their centers are combined and are located at the point O. We align the cylindrical system (ρ, φ, z) with this point so that the axis Oz coincides with the symmetry axis of the spheroids and the anisotropy axis. The surface has the equation

$$\frac{\rho^2}{d_{j,2}^2} + \frac{z^2}{d_{j,1}^2} = 1, \quad (37)$$

where $d_{j,1} / d_{j,2} < \min\{\sqrt{\nu_1}, \sqrt{\nu_2}\}$, $i = 1,2$. We will use the notation introduced above.

A feature of a transversely isotropic body occupying the domain Ω is that its boundary cannot be described by only one pair of oblate spheroidal coordinates, even when the surfaces (37) have a common focal disk. Therefore, the problem for the doubly connected domain of the considered geometry cannot be solved by the usual Fourier method, and it is possible to do this only by the generalized Fourier method. To describe the domain Ω, it is convenient to use two pairs of oblate spheroidal coordinate systems $\{(\tilde{\xi}_{j,s}, \tilde{\eta}_{j,s}, \varphi)\}_{s,j=1}^{2}$, related to cylindrical coordinates by the formulas

$$\rho = a_{j,s}\mathrm{ch}\tilde{\xi}_{j,s}\sin\tilde{\eta}_{j,s}, \quad z = \sqrt{\nu_s}\, a_{j,s}\mathrm{sh}\tilde{\xi}_{j,s}\cos\tilde{\eta}_{j,s}.$$

Then the surface Γ_j is given by the equation $\tilde{\xi}_{j,s} = \tilde{\xi}_{j,s}^{(0)}$, where $a_{j,s}\mathrm{ch}\tilde{\xi}_{j,s}^{(0)} = d_{j,2}$, $\sqrt{\nu_s}\, a_{j,s}\mathrm{sh}\tilde{\xi}_{j,s}^{(0)} = d_{j,1}$ (the parameters of oblate spheroidal coordinate systems, unlike the previous one, are denoted by $a_{j,s}$, so as not to confuse them with elastic constants).

Let us consider the problem of determining the stress state of the above transversely isotropic body, in which the cavity is free from stresses, and an arbitrary balanced axisymmetric load

$$F\vec{V}\Big|_{\xi_{2,s}=\xi_{2,s}^{(0)}} = \frac{c_{44}}{H_2}\sum_{n=0}^{\infty}\left[b_n^{(1)}P_n^{(1)}\left(\cos\tilde{\eta}_2\right)\vec{\mathbf{e}}_\rho + b_n^{(0)}P_n\left(\cos\tilde{\eta}_2\right)\vec{\mathbf{e}}_z\right], \tag{38}$$

is applied to the outer surface. Here $H_2 = \left(d_{2,1}^2\sin^2\tilde{\eta}_2 + d_{2,2}^2\cos^2\tilde{\eta}_2\right)^{1/2}$.

The conditions for the coefficients of the series (38) are given in (26). From a mathematical point of view, the problem is reduced to solving the boundary value problem for the system of equations (1), (2) in the domain Ω with the boundary conditions (38) and

$$F\vec{V}\Big|_{\xi_{1,s}=\xi_{1,s}^{(0)}} = 0. \tag{39}$$

Construction of a general solution to the problem and its reduction to a resolving system. We will search for a general solution to the problem (1), (2), (38), (39) in the domain Ω in the form

$$\vec{V}(\rho, z) = \sum_{s=1}^{2}\sum_{n=0}^{\infty}A_{s,n}^{(1)}\vec{V}_{s,n}^{+(6)}(\tilde{\xi}_{1,s}, \tilde{\eta}_{1,s}) + \sum_{s=1}^{2}\sum_{n=1}^{\infty}A_{s,n}^{(2)}\vec{V}_{s,n}^{-(6)}(\tilde{\xi}_{2,s}, \tilde{\eta}_{2,s}), \tag{40}$$

where $A_{s,n}^{(j)}$ are the unknown coefficients that need to be found in the process of solving the problem. Here, the equilibrium of the load on the external surface of the body has already been taken into account according to Theorem 2. We will solve the problem using the generalized Fourier method. For this, we will use the theorems of addition of basic solutions (4), which are consequences of the general formulas obtained in [11]

$$\vec{V}_{s,n}^{-(6)}(\tilde{\xi}_{2,s}, \tilde{\eta}_{2,s}) = \sum_{k=0}^{n} g_{n,k}^{-(66)}(a_{1,s}, a_{2,s})\vec{V}_{s,k}^{-(6)}(\tilde{\xi}_{1,s}, \tilde{\eta}_{1,s}), \tag{41}$$

$$\vec{V}_{s,n}^{+(6)}(\tilde{\xi}_{1,s}, \tilde{\eta}_{1,s}) = \sum_{k=0}^{\infty} g_{n,k}^{+(66)}(a_{1,s}, a_{2,s})\vec{V}_{s,k}^{+(6)}(\tilde{\xi}_{2,s}, \tilde{\eta}_{2,s}), \quad a_{2,s}\mathrm{sh}\tilde{\xi}_{2,s} > a_{1,s}, \tag{42}$$

where

$$g_{n,k}^{-(66)}(a_{1,s}, a_{2,s}) = \sum_{p=k}^{n}\left(\frac{a_{1,s}}{a_{2,s}}\right)^{p}\frac{i^{n-p}\varepsilon_{n,p}\varepsilon_{p,k}\left(k+1/2\right)\Gamma(n/2+p/2+1/2)}{\Gamma(n/2-p/2+1)\Gamma\left(p/2-k/2+1\right)\Gamma\left(p/2+k/2+3/2\right)}, \tag{43}$$

$$g_{n,k}^{+(66)}(a_{1,s}, a_{2,s}) = \sum_{p=0}^{\infty}\frac{i^{k-p}\varepsilon_{p,n}\varepsilon_{p,k}\left(k+1/2\right)\Gamma\left(p/2+k/2+1/2\right)}{\Gamma(p/2-n/2+1)\Gamma(p/2+n/2+3/2)\Gamma\left(k/2-p/2+1\right)}\left(\frac{a_{1,s}}{a_{2,s}}\right)^{p+1}, \tag{44}$$

$$\varepsilon_{n,k} = \begin{cases} 1, & n-k = 2p,\ p\in\mathbb{Z}, \\ 0, & n-k = 2p+1,\ p\in\mathbb{Z}, \end{cases}$$

$\Gamma(x)$ is Euler's gamma function.

Let us transform the displacement vector $\vec{V}(\rho, z)$ to each separate coordinate system using formulas (3.5), (3.6). As a result, we have

$$\vec{V}(\tilde{\xi}_{1,s}, \tilde{\eta}_{1,s}) = \sum_{s=1}^{2}\sum_{n=0}^{\infty}A_{s,n}^{(1)}\vec{V}_{s,n}^{+(6)}(\tilde{\xi}_{1,s}, \tilde{\eta}_{1,s}) + \sum_{s=1}^{2}\sum_{n=0}^{\infty}\vec{V}_{s,n}^{-(6)}(\tilde{\xi}_{1,s}, \tilde{\eta}_{1,s})\sum_{k=n}^{\infty}g_{k,n}^{-(66)}(a_{1,s}, a_{2,s})A_{s,k}^{(2)}, \tag{45}$$

$$\vec{V}(\tilde{\xi}_{2,s}, \tilde{\eta}_{2,s}) = \sum_{s=1}^{2}\sum_{n=1}^{\infty}A_{s,n}^{(2)}\vec{V}_{s,n}^{-(6)}(\tilde{\xi}_{2,s}, \tilde{\eta}_{2,s}) + \sum_{s=1}^{2}\sum_{n=0}^{\infty}\vec{V}_{s,n}^{+(6)}(\tilde{\xi}_{2,s}, \tilde{\eta}_{2,s})\sum_{k=0}^{\infty}g_{k,n}^{+(66)}(a_{1,s}, a_{2,s})A_{s,k}^{(1)}. \tag{46}$$

Let us pass in formulas (45), (46) from displacements to stresses on surfaces Γ_1 and Γ_2, using formula (6) for this. After satisfying the boundary conditions, we obtain an infinite system of linear algebraic equations with respect to unknown coefficients

$$\sum_{s=1}^{2} t_{1,s,n}^{+(m)} A_{s,n}^{(1)} + \sum_{s=1}^{2} t_{1,s,n}^{-(m)} \sum_{k=n}^{\infty} g_{k,n}^{-(66)}(a_{1,s}, a_{2,s}) A_{s,k}^{(2)} = 0,\ n \ge 0,\ m = 0;1, \tag{47}$$

$$\sum_{s=1}^{2} t_{2,s,n}^{-(m)} A_{s,n}^{(2)} + \sum_{s=1}^{2} t_{2,s,n}^{+(m)}(\tilde{\xi}_{2,s}, \tilde{\eta}_{2,s}) \sum_{k=0}^{\infty} g_{k,n}^{+(66)}(a_{1,s}, a_{2,s}) A_{s,k}^{(1)} = f_n^{(m)},\ n \ge 1,\ m = 0;1, \tag{48}$$

where

$$t_{j,s,n}^{\pm(1)} = \frac{k_s + 1}{\sqrt{\nu_s}} n(n+1) \begin{Bmatrix} Q_n(i\overline{q}_{j,s}) \\ P_n(i\overline{q}_{j,s}) \end{Bmatrix} - \frac{c_{11} - c_{12}}{c_{44}} \frac{d_{j,1}}{d_{j,2}} \begin{Bmatrix} Q_n^{(1)}(i\overline{q}_{j,s}) \\ P_n^{(1)}(i\overline{q}_{j,s}) \end{Bmatrix},\ t_{j,s,n}^{\pm(0)} = (k_s + 1) \begin{Bmatrix} Q_n^{(1)}(i\overline{q}_{j,s}) \\ P_n^{(1)}(i\overline{q}_{j,s}) \end{Bmatrix}, \tag{49}$$

$$f_n^{(1)} = -n(n+1) b_n^{(1)},\ f_n^{(0)} = -b_n^{(0)}.$$

Analysis of the resolving system. Let's transform the system to normal form

$$\tilde{A}_{1,n}^{(1)} + \frac{t_{1,2,n}^{+(0)} Q_n^{(1)}(i\overline{q}_{1,1})}{\Delta_n^{+(2)6}(\tilde{\xi}_{1,1}^0, \tilde{\xi}_{1,2}^0)} \sum_{s=1}^{2} t_{1,s,n}^{-(1)} \sum_{k=n}^{\infty} \frac{g_{k,n}^{-(66)}(a_{1,s}, a_{2,s})}{P_k^{(1)}(i\overline{q}_{2,s})} \tilde{A}_{s,k}^{(2)} - \frac{t_{1,2,n}^{+(1)} Q_n^{(1)}(i\overline{q}_{1,1})}{\Delta_n^{+(2)6}(\tilde{\xi}_{1,1}^0, \tilde{\xi}_{1,2}^0)} \sum_{s=1}^{2} t_{1,s,n}^{-(0)} \sum_{k=n}^{\infty} \frac{g_{k,n}^{-(66)}(a_{1,s}, a_{2,s})}{P_k^{(1)}(i\overline{q}_{2,s})} \tilde{A}_{s,k}^{(2)} = 0, \tag{50}$$

$$A_{2,n}^{(1)} + \frac{t_{1,1,n}^{+(1)} Q_n^{(1)}(i\overline{q}_{1,2})}{\Delta_n^{+(2)6}(\tilde{\xi}_{1,1}^0, \tilde{\xi}_{1,2}^0)} \sum_{s=1}^{2} t_{1,s,n}^{-(0)} \sum_{k=n}^{\infty} \frac{g_{k,n}^{-(66)}(a_{1,s}, a_{2,s})}{P_k^{(1)}(i\overline{q}_{2,s})} A_{s,k}^{(2)} - \frac{t_{1,1,n}^{+(0)} Q_n^{(1)}(i\overline{q}_{1,2})}{\Delta_n^{+(2)6}(\tilde{\xi}_{1,1}^0, \tilde{\xi}_{1,2}^0)} \sum_{s=1}^{2} t_{1,s,n}^{-(1)} \sum_{k=n}^{\infty} \frac{g_{k,n}^{-(66)}(a_{1,s}, a_{2,s})}{P_k^{(1)}(i\overline{q}_{2,s})} A_{s,k}^{(2)} = 0, \tag{51}$$

$$A_{1,n}^{(2)} + \frac{t_{2,2,n}^{-(0)} P_n^{(1)}(i\overline{q}_{2,1})}{\Delta_n^{-(2)6}(\tilde{\xi}_{2,1}^0, \tilde{\xi}_{2,2}^0)} \sum_{s=1}^{2} t_{2,s,n}^{+(1)} \sum_{k=0}^{\infty} \frac{g_{k,n}^{+(66)}(a_{1,s}, a_{2,s})}{Q_k^{(1)}(i\overline{q}_{1,s})} A_{s,k}^{(1)} - \frac{t_{2,2,n}^{-(1)} P_n^{(1)}(i\overline{q}_{2,1})}{\Delta_n^{-(2)6}(\tilde{\xi}_{2,1}^0, \tilde{\xi}_{2,2}^0)} \sum_{s=1}^{2} t_{2,s,n}^{+(0)} \sum_{k=0}^{\infty} \frac{g_{k,n}^{+(66)}(a_{1,s}, a_{2,s})}{Q_k^{(1)}(i\overline{q}_{1,s})} A_{s,k}^{(1)} =$$

$$= \frac{P_n^{(1)}(i\overline{q}_{2,1}) t_{2,2,n}^{-(0)} f_n^{(1)}}{\Delta_n^{-(2)6}(\tilde{\xi}_{2,1}^0, \tilde{\xi}_{2,2}^0)} - \frac{P_n^{(1)}(i\overline{q}_{2,1}) t_{2,2,n}^{-(1)} f_n^{(0)}}{\Delta_n^{-(2)6}(\tilde{\xi}_{2,1}^0, \tilde{\xi}_{2,2}^0)}, \tag{52}$$

$$A_{2,n}^{(2)} + \frac{t_{2,1,n}^{-(1)} P_n^{(1)}(i\overline{q}_{2,2})}{\Delta_n^{-(2)6}(\tilde{\xi}_{2,1}^0, \tilde{\xi}_{2,2}^0)} \sum_{s=1}^{2} t_{2,s,n}^{+(0)} \sum_{k=0}^{\infty} \frac{g_{k,n}^{+(66)}(a_{1,s}, a_{2,s})}{Q_k^{(1)}(i\overline{q}_{1,s})} \tilde{A}_{s,k}^{(1)} - \frac{t_{2,1,n}^{-(0)} P_n^{(1)}(i\overline{q}_{2,2})}{\Delta_n^{-(2)6}(\tilde{\xi}_{2,1}^0, \tilde{\xi}_{2,2}^0)} \sum_{s=1}^{2} t_{2,s,n}^{+(1)} \sum_{k=0}^{\infty} \frac{g_{k,n}^{+(66)}(a_{1,s}, a_{2,s})}{Q_k^{(1)}(i\overline{q}_{1,s})} \tilde{A}_{s,k}^{(1)} =$$

$$= \frac{P_n^{(1)}(i\overline{q}_{2,2}) t_{2,1,n}^{-(1)} f_n^{(0)}}{\Delta_n^{-(2)6}(\tilde{\xi}_{2,1}^0, \tilde{\xi}_{2,2}^0)} - \frac{P_n^{(1)}(i\overline{q}_{2,2}) t_{2,1,n}^{-(0)} f_n^{(1)}}{\Delta_n^{-(2)6}(\tilde{\xi}_{2,1}^0, \tilde{\xi}_{2,2}^0)}, \tag{53}$$

where

$$A_{s,n}^{(1)} = \tilde{A}_{s,n}^{(1)} / Q_n^{(1)}(i\overline{q}_{1,s}),\ A_{s,n}^{(2)} = \tilde{A}_{s,n}^{(2)} / P_n^{(1)}(i\overline{q}_{2,s}).$$

Theorem 3. *If conditions*

$$a_{1,s} \operatorname{sh} \tilde{\xi}_{1,s}^{(0)} < \begin{cases} a_{2,s} \operatorname{sh} \tilde{\xi}_{2,s}^{(0)},\ a_{1,s} < a_{2,s}, \\ a_{2,s} \operatorname{ch} \tilde{\xi}_{2,s}^{(0)},\ a_{1,s} > a_{2,s}, \end{cases} \quad a_{2,s} \operatorname{sh} \tilde{\xi}_{2,s}^{(0)} > \begin{cases} a_{1,s} \operatorname{sh} \tilde{\xi}_{1,s}^{(0)},\ a_{2,s} < a_{1,s}, \\ a_{1,s} \operatorname{ch} \tilde{\xi}_{1,s}^{(0)},\ a_{2,s} > a_{1,s} \end{cases} \tag{54}$$

are satisfied, the system (50) – (53) has a Fredholm operator in the space l_2^4.

Proof of the theorem.

To prove the theorem, it is sufficient to show the convergence of the series

$$\sum_{s=1}^{2} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left| \frac{t_{2,p,n}^{-(j)} P_n^{(1)}(i\overline{q}_{2,3-p})}{\Delta_n^{-(2)6}(\tilde{\xi}_{2,1}^0, \tilde{\xi}_{2,2}^0)} \frac{t_{2,s,n}^{+(1-j)} g_{k,n}^{+(66)}(a_{1,s}, a_{2,s})}{Q_k^{(1)}(i\overline{q}_{1,s})} \right|^2 < \infty, \tag{55}$$

$$\sum_{s=1}^{2} \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \left| \frac{t_{1,p,n}^{+(j)} Q_n^{(1)}(i\overline{q}_{1,3-p})}{\Delta_n^{+(2)6}(\tilde{\xi}_{1,1}^0, \tilde{\xi}_{1,2}^0)} \frac{t_{1,s,n}^{-(1-j)} g_{k,n}^{-(66)}(a_{1,s}, a_{2,s})}{P_k^{(1)}(i\overline{q}_{2,s})} \right|^2 < \infty, \tag{56}$$

where $j = 0, 1$; $p, s = 1, 2$.

First, we note that thanks to estimates (11), (17), (25), (33), we can estimate the factors of the common terms of the series (55), (56) as follows:

$$\left| \frac{t_{2,p,n}^{-(j)} P_n^{(1)}(i\overline{q}_{2,3-p})}{\Delta_n^{-(2)6}(\tilde{\xi}_{2,1}^{(0)}, \tilde{\xi}_{2,2}^{(0)})} \right| < C^- n^j,\ \left| \frac{t_{1,p,n}^{+(j)} Q_n^{(1)}(i\overline{q}_{1,3-p})}{\Delta_n^{+(2)6}(\tilde{\xi}_{1,1}^0, \tilde{\xi}_{1,2}^0)} \right| < C^+ n^j,\ j = 0, 1. \tag{57}$$

From formula (42) follows the Parseval equality

$$\int_0^{\pi} \left| Q_n(i \operatorname{sh} \tilde{\xi}_{1,s}) P_n(\cos \tilde{\eta}_{1,s}) \right|^2 \sin \tilde{\eta}_{2,s} d\tilde{\eta}_{2,s} = \sum_{k=0}^{\infty} \left| g_{n,k}^{+(66)}(a_{1,s}, a_{2,s}) Q_k(i \operatorname{sh} \tilde{\xi}_{2,s}) \right|^2 \frac{2}{2k+1}$$

thanks to which we can record

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|g_{k,n}^{+(66)}(a_{1,s},a_{2,s})\frac{Q_n(i\,\mathrm{sh}\tilde{\xi}_{2,s})}{Q_k^{(1)}(i\bar{q}_{1,s})}\right|^2\frac{2}{2n+1}=\int_0^{\pi}\sum_{k=0}^{\infty}\left|\frac{Q_k(i\,\mathrm{sh}\tilde{\xi}_{1,s})}{Q_k^{(1)}(i\bar{q}_{1,s})}P_k(\cos\tilde{\eta}_{1,s})\right|^2\sin\tilde{\eta}_{2,s}d\tilde{\eta}_{2,s}\ . \qquad (58)$$

Let us emphasize that in Parseval's formula $(\tilde{\xi}_{1,s},\tilde{\eta}_{1,s})$ and $(\tilde{\xi}_{2,s},\tilde{\eta}_{2,s})$ are the coordinates of the same point. Let us substitute into the formula (58) $\tilde{\xi}_{2,s}=\tilde{\xi}_{2,s}^{(0)}$. In this case, the coordinate $\tilde{\xi}_{1,s}$ on the fixed surface is some function $\tilde{\xi}_{1,s}=\tilde{\xi}_{1,s}(\tilde{\xi}_{2,s}^{(0)},\tilde{\eta}_{2,s})$. The condition for the convergence of the series under the integral (58) due to the estimates (17), (20) is the inequality

$$\tilde{\xi}_{1,s}(\tilde{\xi}_{2,s}^{(0)},\tilde{\eta}_{2,s})>\tilde{\xi}_{1,s}^{(0)}\quad\forall\tilde{\eta}_{2,s}\in[0,\pi]$$

or, which is the same,

$$\min_{\tilde{\eta}_{2,s}\in[0,\pi]}\tilde{\xi}_{1,s}(\tilde{\xi}_{2,s}^{(0)},\tilde{\eta}_{2,s})>\tilde{\xi}_{1,s}^{(0)}\ . \qquad (59)$$

Let us find the minimum value of the coordinate in (59). From the formulas for the connection of cylindrical and spheroidal coordinates it follows that

$$\left(\mathrm{ch}\tilde{\xi}_{2,s}^{(0)}\sin\tilde{\eta}_{2,s}/\mathrm{ch}\tilde{\xi}_{1,s}\right)^2+\left(\mathrm{sh}\tilde{\xi}_{2,s}^{(0)}\cos\tilde{\eta}_{2,s}/\mathrm{sh}\tilde{\xi}_{1,s}\right)^2=\left(a_{1,s}/a_{2,s}\right)^2,$$

whence

$$\mathrm{sh}^2\tilde{\xi}_{1,s}=(q_{2,s}^2-\cos^2\tilde{\eta}_{2,s}-\alpha^2+\sqrt{(q_{2,s}^2-\cos^2\tilde{\eta}_{2,s}-\alpha^2)^2+4\alpha^2\bar{q}_{2,s}^2\cos^2\tilde{\eta}_{2,s}})/(2\alpha^2)\,, \qquad (60)$$

where $\alpha=a_{1,s}/a_{2,s}$. Let us examine the right-hand side of formula (60) at the extremum. It can be shown that for $\alpha<1$

$$\mathrm{sh}^2\tilde{\xi}_{1,s}^{\min}=\bar{q}_{2,s}^2/\alpha^2\,. \qquad (61)$$

In the same way for $\alpha>1$

$$\mathrm{sh}^2\tilde{\xi}_{1,s}^{\min}=q_{2,s}^2/\alpha^2\,. \qquad (62)$$

Now the consequences of formulas (59), (61), (62) are the first convergence condition (54).

Therefore, by the first condition (54), the series on the left in (58) converges. It is not difficult to show that the convergence remains for any additional factors in this series of the form n^p, k^q $(p,q>0)$. Therefore, taking into account the estimate (57), we obtain (55).

For the series (56), a similar approach leads to the second convergence condition (54). Thus, the theorem is proved.

Remark: The convergence conditions of method (54) can be given the following geometric meaning:

$$d_{1,1}<\begin{cases}d_{2,1},\ a_{1,s}<a_{2,s},\\ \sqrt{\nu_s}d_{2,2},\ a_{1,s}>a_{2,s},\end{cases}\quad d_{2,1}>\begin{cases}d_{1,1},\ a_{2,s}<a_{1,s},\\ \sqrt{\nu_s}d_{1,2},\ a_{2,s}>a_{1,s}.\end{cases}$$

The last formulas show that not only do the surfaces Γ_1 and Γ_2, as geometric objects, not intersect, but also do not intersect the surfaces after the similarity transformation along the axis Oz with the coefficient $1/\sqrt{\nu_s}$.

The problem of a transversely isotropic oblate spheroid with a circular crack. Let us illustrate the above results by the example of the numerical solution of the axisymmetric problem of the stress state of a transversely isotropic oblate spheroid Γ_2 (37) with a circular crack $\Gamma_1=\{(\rho,z):z=0,\ \rho\in[0,a]\}$, which is under the action of a constant normal load. The boundary conditions at the crack edges are set as follows:

$$\sigma_{z|\Gamma_1}=-\sigma\,,\ \ \tau_{\rho z|\Gamma_1}=0\,,\ \ \tau_{\varphi z|\Gamma_1}=0\,, \qquad (63)$$

where σ_z, $\tau_{\rho z}$, $\tau_{\varphi z}$ are components of the stress tensor in cylindrical coordinates. The outer surface of the spheroid is considered free from forces.

If we look for a solution to this problem in the form (40), then the unknown coefficients must satisfy the system (47), (48) with the other right-hand sides:

$$\sum_{s=1}^{2}t_{1,s,n}^{+(m)}A_{s,n}^{(1)}+\sum_{s=1}^{2}t_{1,s,n}^{-(m)}\sum_{k=n}^{\infty}g_{k,n}^{-(66)}(a,a_{2,s})A_{s,k}^{(2)}=f_n^{(m)}\,,\ \ n\ge 0\,,\ \ m=0;1\,, \qquad (64)$$

$$\sum_{s=1}^{2}t_{2,s,n}^{-(m)}A_{s,n}^{(2)}+\sum_{s=1}^{2}t_{2,s,n}^{+(m)}\sum_{k=0}^{n}g_{k,n}^{+(66)}(a,a_{2,s})A_{s,k}^{(1)}=0\,,\ \ n\ge 1\,,\ \ m=0;1\,, \qquad (65)$$

where the coefficients for the unknowns are described by formulas (49), into which we must substitute $\bar{q}_{1,s}=0$, and the right-hand side is $f_n^{(m)}=\delta_{m,0}\delta_{n,1}\sigma a/c_{44}$.

After numerically solving the system (64), (65) by the reduction method, it is possible to analyze, for example, the normal stresses that arise in the crack plane outside its boundary and are given by the formula

$$\sigma_z(\rho,0)_{|\rho>a} = \frac{c_{44}}{\sqrt{\rho^2-a^2}}\sum_{s=1}^{2}(k_s+1)\sum_{n=0}^{\infty}A_{s,n}^{(1)}Q_n\left(i\sqrt{(\rho/a)^2-1}\right)P_n^{(1)}(0)-$$

$$-c_{44}\sum_{s=1}^{2}\frac{k_s+1}{\sqrt{a_{2,s}^2-\rho^2}}\sum_{n=1}^{\infty}A_{s,n}^{(2)}P_n^{(1)}(i0)P_n\left(\sqrt{1-(\rho/a_{2,s})^2}\right). \qquad (66)$$

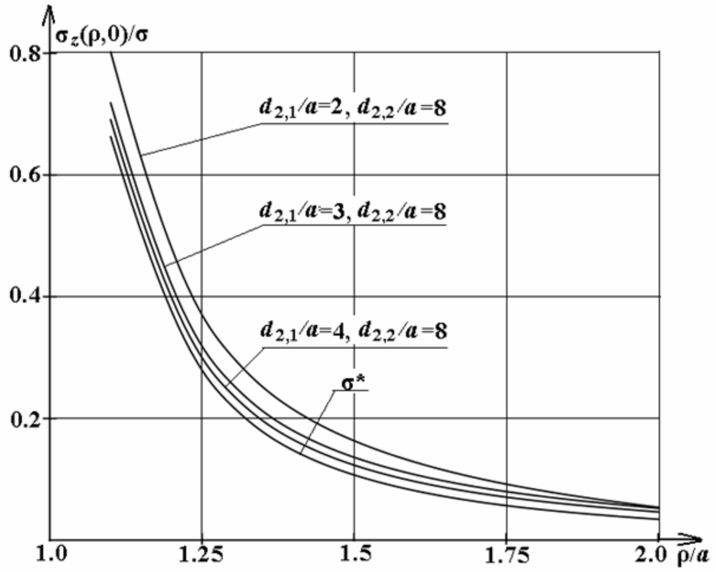

Fig. 1 – Distribution of normal stresses in the plane of the crack beyond its boundary.

Fig. 1 shows graphs of normal stresses $\sigma_z(\rho,0)_{|\rho>a}/a$ (66) depending on the relative sizes of the semiaxes of the spheroid. Here, the symbol σ^* denotes the stresses that correspond to a crack in transversely isotropic space (the problem has an analytical solution in closed form). The graphs show that for a fixed size of the semi-major axis of the spheroid, the stresses increase as the size of the semi-minor axis decreases. A different situation is observed for a fixed size of the semi-minor axis. Changing the size of the semi-major axis has practically no effect on the magnitude of the stresses.

Let us also write down the formula for calculating the stress intensity factor at the crack boundary:

$$K_{1z} = \lim_{\rho\to a+0}\sqrt{2\pi(\rho-a)}\sigma(\rho,0).$$

Transition in system (64), (65) to dimensionless unknowns by the formula

$$A_{s,n}^{(j)} = \frac{\sigma a}{c_{44}}\tilde{A}_{s,n}^{(j)}$$

leads to the following formula for the SIF:

$$K_{1z}/(\sqrt{a}\sigma) = \sqrt{\pi}\sum_{s=1}^{2}(k_s+1)\sum_{k=1}^{\infty}\tilde{A}_{s,2k-1}^{(1)}. \qquad (67)$$

Table 1 – SIF values depending on the relative sizes of the semiaxes of the spheroid

$d_{2,1}/a \setminus d_{2,2}/a$	6.0	7.0	8.0
2.0	1.2382	1.2356	1.2336
3.0	1.1715	1.1694	1.1681
4.0	1.1506	1.1489	1.1479

Table 1 shows the values of the stress intensity factor depending on the relative sizes of the semiaxes of the spheroid. A decrease in the SIF is observed both at a fixed size of the major semiaxis and an increase in the size of the minor, and at a fixed size of the minor semiaxis and an increase in the size of the major. For comparison, we present the SIF of a crack in transversely isotropic space

$$K_{1z}/(\sqrt{a}\sigma) = 2/\sqrt{\pi} \approx 1.1284.$$

Table 2 – Practical convergence of the reduction method

$n_{\max} \setminus \rho / a$	1.1	1.3	1.5	1.7	1.9
20	0.8021	0.2879	0.1608	0.1015	0.0665
30	0.8019	0.2873	0.1602	0.1012	0.0665
40	0.8019	0.2873	0.1602	0.1012	0.0665

Table 2 shows the practical convergence of the reduction method using the example of stress values $\sigma_z(\rho,0)\setminus\sigma$ at $d_{2,1}/a = 2.0$, $d_{2,2}/a = 8.0$, The number $n_{\max}$ specifies the reduction parameter at which the size of the reduced system is equal to $4n_{\max} \times 4n_{\max}$.

Prospects for further research. One of the directions of further research is the substantiation of solutions to the equilibrium equations of transversally isotropic canonical bodies.

Conclusions. For the first time, an exact, substantiated solution by the Fourier method of the second axisymmetric boundary value problems of the theory of elasticity in the general formulation for a transversely isotropic oblate spheroid and a space with a spheroidal cavity has been obtained. Similar problems are relevant in local modeling of the stress state of transversely isotropic bodies with cavities, inclusions, and cracks. The review of previous research on this topic presented in the article showed that the problems of substantiating the basic boundary value problems for the above-mentioned transversally isotropic bodies were not posed and not solved. However, ignoring this problem can lead to incorrect results even in the works of classics, as shown by the authors of the study [6]. A fundamental problem of justification, which could not be solved for many years, was the problem of estimating from below the moduli of determinants of resolving systems for interior and exterior problems. The indicated estimates were obtained in this work. The complexity of estimating determinants is due to the fact that they depend on nine parameters that are functionally related to each other, and in addition, two of them are included in the arguments of Legendre functions of the first and second kind. The estimates found made it possible to formulate and prove theorems about the solvability conditions of the considered boundary value problems in certain classes of functions. The obtained results are applied to the solution of the second boundary value problem for a transversely isotropic oblate spheroid with a spheroidal cavity, the centers and directions of the axes of which coincide. An arbitrary symmetric balanced load is given on the surfaces of the spheroid, which satisfies a certain condition for the convergence of the series of limit functions developed in terms of Legendre functions. A feature of this problem for a transversely isotropic body, is the impossibility of describing spheroidal surfaces with any geometry by a single pair of spheroidal coordinate systems. This means that such a problem can be solved not by the usual, but by the generalized Fourier method. Its application made it possible to reduce the original problem to an infinite system of linear algebraic equations. Thanks to the above estimates of the determinants of simply connected problems, the Fredholm property of the system operator in a certain Hilbert space has been proven. Numerical results in the considered problem were obtained in the case of a oblate spheroid with a circular crack. The surface of the spheroid is assumed to be free of forces, and a constant normal load is applied to the crack. The graphs of normal stresses in the crack plane outside its boundary, as well as the values of the stress intensity factors at its boundary, are presented. A parametric analysis of stresses and SIFs depending on the geometric parameters of the problem is carried out. The practical convergence of the reduction method in solving an infinite system is investigated.

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