UDC 517.968 **DOI: 10.20998/2222-0631.2025.01(8).11**

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SOLVING SOME COMPLETE SINGULAR INTEGRAL EQUATIONS BY INTEGRAL TRANSFORMATIONS AND ANALYTIC EXTENSIONS

In this paper, the solution of some singular integral equations is presented. The coefficients of the equations (external and internal) possess pointwise weak singularities on the integration interval, and furthermore, the extension of these coefficients over the entire complex plane yields multivalued functions. To obtain certain analytical branches, cuts are made in the complex plane, so that at the edges of the cuts, the extracted branches take different values. Certain integral (equivalent) transformations are carried out on the operator defined by the given equation, and after a series of mathematical reasonings and calculations, the given equation is reduced to a characteristic singular integral equation, studied in the monographs of academicians N. Muskhvelishvili and F.Gahov. The solutions obtained are in strict accordance with the results presented in the aforementioned monographs.

Key words: singular integral equation, Riemann boundary value problem, multivalued functions, analytic continuation, residue, Cauchy type integral.

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РОЗВ'ЯЗУВАННЯ ДЕЯКИХ ПОВНИХ СИНГУЛЯРНИХ ІНТЕГРАЛЬНИХ РІВНЯНЬ ІНТЕГРАЛЬНИМИ ПЕРЕТВОРЕННЯМИ ТА АНАЛІТИЧНИМИ РОЗШИРЕННЯМИ

У даній роботі представлено рішення деяких сингулярних інтегральних рівнянь. Коефіцієнти рівнянь (зовнішні та внутрішні) мають поточково слабкі особливості на інтервалі інтегрування, крім того, продовження цих коефіцієнтів на всю комплексну площину represents multiform functions. Для отримання певних аналітичних гілок розрізи роблять у комплексній площині, так що на краях розрізів вилучені гілки набувають різних значень. Над оператором, визначеним даним рівнянням, здійснюються певні інтегральні (еквівалентні) перетворення, які після серії математичних міркувань і обчислень зводять дане рівняння до характерного сингулярного інтегрального рівняння, дослідженого в монографіях академіків Н. Мусхвелішвілі та Ф. Гахова. Отримані рішення чітко узгоджуються з результатами, представленими у згаданих монографіях.

Ключові слова: сингулярне інтегральне рівняння, крайова задача Рімана, багатозначні функції, аналітичне продовження, відрахування, інтеграл типу Коші.

Introduction. Integral equations are found in various fields of science and in numerous applications (such as physics, control theory, economics, and medicine). The theory of such equations was founded by *A. Poincaré* and *D. Hilbert* almost immediately after the appearance of *the classical theory of Fredholm integral equations*. Exact solutions of integral equations play a major role in the formation of a correct understanding of the qualitative characteristics of many phenomena and processes in various fields of natural sciences. However, as mentioned in the monographs of *N. Muskelishvili* and *F. Gahov* and in other works, the solution of singular integral equations can be determined in rare cases, and even in these cases, the determination of an exact solution requires the calculation of singular integrals, which is accompanied by great difficulties both theoretical and computational. The most often applied method for solving singular integral equations on closed contours consists in the equivalent reduction of the given equation to *the Riemann-type boundary value problem*, which is solved effectively. In the case of an open integration contour (bounded or unbounded segment) this method can no longer be applied and consequently various other methods are applied depending on the given equation.

In this paper, the method of analytical extension over the entire complex plane of the unknown function and the coefficients of the equation, of some integral transformations, as well as the theory of residues and the passage to the limit is applied so that the given equation reduces to a characteristic equation. The obtained results are compared with some known results for some equations that represent particular cases of the equations considered in this paper. As a result of these comparisons, we are convinced that the obtained results are correct.

L. Levin [1] proved that in the case when $a(x)=\left(1-x^2\right)^{\frac{1}{2}}$, $b(x)=\lambda^2\left(1-x^2\right)^{\frac{1}{2}}$ and λ is constant, the singular integral equation

$$a(x)\frac{1}{\pi i}\int_{-1}^{1}\frac{\varphi(t)}{t-x}dt+\frac{1}{\pi i}\int_{-1}^{1}\frac{b(t)}{t-x}\varphi(t)dt=f(x) \tag{1}$$

can be reduced to solving two independent *Carleman integral equations* which are solved in the usual way. For a somewhat more general case, when

$$a(x)b(x)=\lambda^2\left(1-x^2\right),$$

equation (1) was solved in [2, 3].

In the fourth paragraph of this work, complete integral equations of the form are solved

$$\frac{1}{\pi i}\int_{a}^{b}\frac{\varphi(t)}{t-x}dt-\lambda^2\frac{1}{\pi i}\int_{a}^{b}\frac{K_m(t,x)}{t-x}\varphi(t)dt=f(x)\ \ (m=1,2,3,4), \tag{2}$$

where λ is a complex parameter, f is a given *Hölder function on the interval* $[a, b]$:

$$K_1(t, x) = \sqrt{\frac{(b-x)(x-a)}{(b-t)(t-a)}}, \quad K_2(t, x) = \sqrt{\frac{(b-x)(t-a)}{(b-t)(x-a)}},$$

$$K_3(t, x) = K_2^{-1}(t, x) \text{ and } K_4(t, x) = K_1^{-1}(t, x).$$

We will consider in detail the cases when $m = 1$ in (2).

The solution is carried out by analytical continuation of the coefficients (multivalued) of the equation, identifying single-valued branches and using some integral transformations. As a result of these actions, the solution of the original equation is reduced to the solution of the usual characteristic singular equation with *the Cauchy kernel*, followed by the application of the theory of the Riemann boundary value problem. In this case, *the Poincaré-Bertrand permutation formula* also plays a significant role:

$$\frac{1}{\pi i}\int_L \frac{d\tau}{\tau - t}\frac{1}{\pi i}\int_L \frac{\varphi(\tau, s)}{s-\tau}ds = \varphi(t, t) + \frac{1}{\pi i}\int_L ds \frac{1}{\pi i}\int_L \frac{\varphi(\tau, s)}{(s-\tau)(\tau - t)}d\tau .$$

The second and third paragraphs provide the necessary information about the relationship between the solution of the singular equation and the solution of the corresponding Riemann boundary value problem. In the fourth paragraph, the explicit solution of the characteristic singular equation is expressed through the solution of the corresponding Riemann boundary value problem.

Relationship between the solution of a singular equation and the solution of the Riemann boundary value problem. Let's consider the simplest type of special integral equation – the characteristic equation:

$$a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}d\tau = f(t). \tag{3}$$

In this case, the solution to the equation can be reduced to the solution to the Riemann boundary value problem, providing a closed-form expression for the solution.

Let us introduce a piecewise analytic function defined by *a Cauchy type integral*,

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z}d\tau, \tag{4}$$

where φ is the desired solution to the characteristic equation.

According to *the Sokhotsky's formulas* [4, 5]

$$\begin{cases} \varphi(t) = \Phi^+(t) - \Phi^-, \\ \frac{1}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}d\tau = \Phi^+(t) + \Phi^-(t). \end{cases} \tag{5}$$

Introducing the values

$$\varphi(t) \text{ and } \frac{1}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}d\tau$$

into equation (3) and solving it for $\Phi^+(t)$, we find that the piecewise analytic function $\Phi(z)$ must be a solution to the Riemann boundary value problem

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \tag{6}$$

Where

$$G(t) = \frac{a(t) - b(t)}{a(t) + b(t)}, \quad g(t) = \frac{f(t)}{a(t) + b(t)}. \tag{7}$$

Since the fact that the desired function $\Phi(z)$ is represented by a Cauchy integral, it must also satisfy the additional condition

$$\Phi^-(\infty) = 0. \tag{8}$$

The index of the coefficient $\frac{a(t) - b(t)}{a(t) + b(t)}$ of the Riemann problem (6) will be called the index of the integral equation (3).

Having solved the boundary value problem (4), we use formula (5) to obtain a solution to equation (3).

Thus, the integral equation (3) has been reduced to the Riemann boundary value problem (6). To establish the equivalence between the equation and the boundary value problem, it is necessary to prove that, conversely, the function

$\varphi(t)$ obtained from the solution of the boundary value problem also satisfies equation (3). To do this, we need to confirm that the second formula in (5) is also valid. Let's prove it. If the solution to problem (6) is represented by a Cauchy-type integral, then both formulas (5) hold, and one can uniquely recover the original equation from the boundary value problem. Now suppose there exists another function $\Phi_1(z)$, that satisfies the same conditions. Then, for the difference

$$\Phi_2(z) = \Phi(z) - \Phi_1(z),$$

the following equality holds:

$$\Phi_2^+(t) - \Phi_2^- = 0 .$$

By the theorem of analytic continuation theorem and *Liouville's theorem* (taking into account (8)) $\Phi_2(z) \equiv 0$. Consequently, $\Phi_1(z) = \Phi(z)$, which completes the proof.

To simplify further formulas, we first divide the entire equation (3) by $\sqrt{a^2(t) - b^2(t)}$, i.e. we assume that the coefficients of this equation satisfy the condition

$$a^2(t) - b^2(t) = 1 .$$

Substituting the limit values of the Cauchy type integral into the boundary condition (6), we obtain *the characteristic singular integral equation*

$$\frac{1}{2}\left[1 + G(t)\right]\varphi(t) + \frac{1 - G(t)}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - t} = g(t) . \quad (9)$$

From the solutions of the last equation using formula (4) we obtain a solution to the Riemann problem.

Solution of the characteristic equation. Let us recall (see [5]) the solution to the Riemann boundary value problem (6), assuming $\chi \geq 0$, and compute the limit values of the corresponding functions using the Sokhotsky formulas

$$\Phi^+(t) = X^+(t)\left[\frac{1}{2}\frac{g(t)}{X^+(t)} + \Psi(t) - \frac{1}{2}P_{\chi-1}(t)\right],$$

$$\Phi^-(t) = X^-(t)\left[-\frac{1}{2}\frac{g(t)}{X^+(t)} + \Psi(t) - \frac{1}{2}P_{\chi-1}(t)\right],$$

where $\Psi(t)$ is the singular integral

$$\Psi(t) = \frac{1}{2\pi i}\int_L \frac{g(\tau)}{X^+(\tau)}\frac{d\tau}{\tau - t} . \quad (10)$$

From here, using formula (5) we obtain

$$\varphi(t) = \frac{1}{2}\left[1 + \frac{X^-(t)}{X^+(t)}\right]g(t) + X^+(t)\left[1 - \frac{X^-(t)}{X^+(t)}\right]\left[\Psi(t) - \frac{1}{2}P_{\chi-1}(t)\right].$$

Based on the boundary condition, we replace $\dfrac{X^-(t)}{X^+(t)} = \dfrac{1}{G(t)}$, and the function $\Psi(t)$ with their respective expressions according to formula (10). Then we have

$$\varphi(t) = \frac{1}{2}\left[1 + \frac{1}{G(t)}\right]g(t) + X^+(t)\left[1 - \frac{1}{G(t)}\right]\left[\frac{1}{2\pi i}\int_L \frac{g(\tau)}{X^+(\tau)}\frac{d\tau}{\tau - t} - \frac{1}{2}P_{\chi-1}(t)\right].$$

If $\chi < 0$, then, as is known (see [5]), the Riemann problem (6), generally speaking, is unsolvable. The condition for its solvability

$$\int_L \frac{g(\tau)}{X^+(\tau)}\tau^{k-1}d\tau = 0 \quad (k = 1, 2, ..., -\chi) \quad (11)$$

will simultaneously serve as the conditions for the solvability of equation (3).

Let us summarize the results of the study.

1^0. If $\chi > 0$, then the homogeneous equation

$$a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}d\tau = 0$$

has χ linearly independent solutions.

2^0 . If $\chi \le 0$, then the homogeneous equation has only then trivial solution.

3^0 . If $\chi > 0$, then the inhomogeneous equation

$$a(t)\varphi(t)+\frac{b(t)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau-t}d\tau = f(t)$$

is solvable for any right-hand side $f(t)$ and its general solution depends linearly on ϰ arbitrary constants.

4^0 . If $\chi < 0$ then the inhomogeneous equation is solvable if and only if its right-hand side $f(t)$ satisfies the $-\chi$ conditions

$$\int_L h_k(t) f(t) d\tau = 0 ,$$

where

$$h_k(t)=\frac{1}{Z(t)}t^{k-1} \quad (k=1,2,...,-\chi) .$$

Solution of equation (2). The operator defined by the left-hand side of Eq. (2) (for $m=1$), is acted upon by the operator

$$(A\psi)(x)=\frac{1}{\pi i}\int_a^b \left(\frac{b-t}{t-a}\right)^p \frac{\psi(t)}{t-x}dt ,$$

where p is some complex number, the choice of which we can control.

In the obtained repeated singular integrals, we change the order of integration, using *the Poincaré-Bertrand formula* [4, 5]. We get

$$\left(1-\lambda^2\right)\left(\frac{b-x}{x-a}\right)^p \varphi(x)+\frac{1}{\pi^2}\int_a^b \varphi(s)ds\int_a^b \left(\frac{b-t}{t-a}\right)^p \frac{dt}{(t-s)(t-x)}dt -$$
$$-\frac{\lambda^2}{\pi^2}\int_a^b \frac{\varphi(s)}{\sqrt{(b-s)(s-a)}}ds\int_a^b \left(\frac{b-t}{t-a}\right)^p \frac{\sqrt{(b-t)(t-a)}}{(t-s)(t-x)}dt = (Af)(x) . \tag{12}$$

We denote the inner integrals by $I_1(s,x)$ and $I_2(s,x)$, respectively. Let us calculate $I_2(s,x)$. We split the outer integral from a to $x(s<x)$ and from x to $b(s>x)$ and calculate $I_2(s,x)$, with respect to $s<x$, and then $s>x$. Due to the symmetry of the function $I_2(s,x)$ with respect to s and x, the result will be the same.

We proceed from the multivalued function

$$F_p(z)=\frac{(z-b)^{\frac{1}{2}+p}(z-a)^{\frac{1}{2}-p}}{(z-s)(z-x)} , \tag{13}$$

for which the points $z=a$ and $z=b$ (and only these) are branch points.

Let us make a cut along the segment $[a,b]$ and select an analytic branch of this function. For this, we fix $\arg(t-a)=0$ and $\arg(b-t)=0$ on the upper side of the cut. We denote by $F_p^+(t)$ the value of this function at point t on the upper side of the cut. We have

$$F_p^+(t)=ie^{p\pi i}\frac{(b-t)^{\frac{1}{2}+p}(t-a)^{\frac{1}{2}-p}}{(t-s)(t-x)} . \tag{14}$$

Let us find the value of this branch at the same point t on the lower side of the cut, bypassing the point $z=a$, or $z=b$. Let us denote this value as $F_p^-(t)$. We get

$$F_p^-(t)=-ie^{-p\pi i}\frac{(b-t)^{\frac{1}{2}+p}(t-a)^{\frac{1}{2}-p}}{(t-s)(t-x)} . \tag{15}$$

Since the expansion of the function $F_p(z)$ in a Laurent series in the vicinity of the point at infinity starts from $\frac{1}{z}$, then

$$\operatorname*{res}_{z=\infty} F_p(z) = -1 .$$

We integrate the function $F_p(z)$ along the closed contour L^- shown in the figure below (Fig. 1).

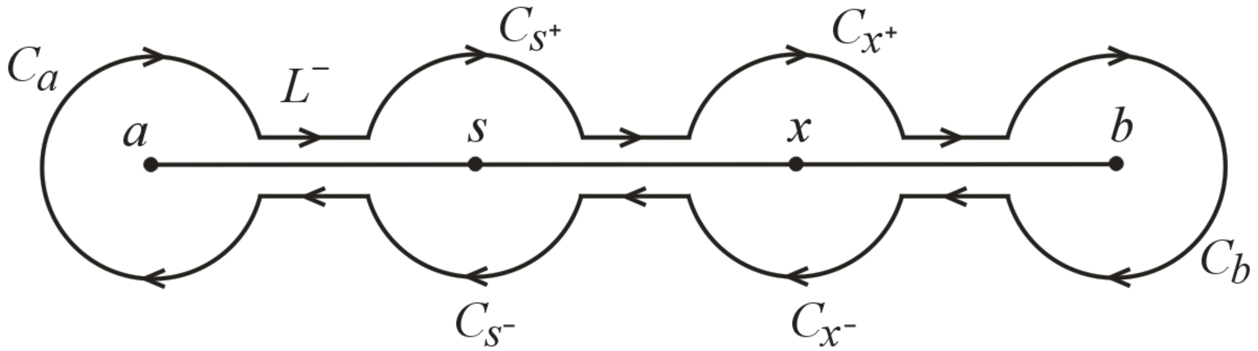

Fig. 1 – Integration contour L^-.

According to the definition of the residue of a function at infinity, we have

$$\int_{L^-} F_p(t)\,dt = 2\pi i \lim_{z=\infty} F_p(z) = -2\pi i . \tag{16}$$

Let ε denote the radius of the circles C_a, C_b and the semicircles C_{s^+}, C_{s^-}, C_{x^+}, C_{x^-}. In equality (16), we take the limit as $\varepsilon \to 0$, assuming that $-\frac{3}{2} < p < \frac{3}{2}$. Due to the weak singularity (infinity of order less than one) of the function $F_p(t)$ at the points $z = a$ and $z = b$, we obtain

$$\lim_{\varepsilon\to 0}\int_{C_a} F_p(t)\,dt = \lim_{\varepsilon\to 0}\int_{C_b} F_p(t)\,dt = 0 .$$

We proceed to calculate the remaining limits, in this case we will use equalities (14) and (15). We get

$$\lim_{\varepsilon\to 0}\int_{C_{s^+}} F_p(t)\,dt = \lim_{\varepsilon\to 0}\int_{\pi}^{0} F_p\left(s+\varepsilon e^{i\varphi}\right) i\varepsilon e^{i\varphi} d\varphi =$$

$$= \lim_{\varepsilon\to 0}\int_{\pi}^{0} \frac{\left(s+\varepsilon e^{i\varphi}-b\right)^{\frac{1}{2}+p}\left(s+\varepsilon e^{i\varphi}-a\right)^{\frac{1}{2}-p}}{\varepsilon e^{i\varphi}\left(s+\varepsilon e^{i\varphi}-x\right)} i\varepsilon e^{i\varphi} d\varphi = i\int_{\pi}^{0}\lim_{\varepsilon\to 0} \frac{\left(s+\varepsilon e^{i\varphi}-b\right)^{\frac{1}{2}+p}\left(s+\varepsilon e^{i\varphi}-a\right)^{\frac{1}{2}-p}}{s+\varepsilon e^{i\varphi}-x} d\varphi =$$

$$= -e^{p\pi i}\frac{(b-s)^{\frac{1}{2}+p}(s-a)^{\frac{1}{2}-p}}{s-x}(-\pi) = \pi e^{p\pi i}\frac{(b-s)^{\frac{1}{2}+p}(s-a)^{\frac{1}{2}-p}}{s-x} . \tag{17}$$

$$\lim_{\varepsilon\to 0}\int_{C_{s^-}} F_p(t)\,dt = \lim_{\varepsilon\to 0}\int_{0}^{-\pi} F_p\left(s+\varepsilon e^{i\varphi}\right) i\varepsilon e^{i\varphi} d\varphi =$$

$$= \lim_{\varepsilon\to 0}\int_{0}^{-\pi} \frac{\left(s+\varepsilon e^{i\varphi}-b\right)^{\frac{1}{2}+p}\left(s+\varepsilon e^{i\varphi}-a\right)^{\frac{1}{2}-p}}{\varepsilon e^{i\varphi}\left(s+\varepsilon e^{i\varphi}-x\right)} i\varepsilon e^{i\varphi} d\varphi = i\int_{0}^{-\pi}\lim_{\varepsilon\to 0} \frac{\left(s+\varepsilon e^{i\varphi}-b\right)^{\frac{1}{2}+p}\left(s+\varepsilon e^{i\varphi}-a\right)^{\frac{1}{2}-p}}{s+\varepsilon e^{i\varphi}-x} d\varphi =$$

$$= e^{-p\pi i}\frac{(b-s)^{\frac{1}{2}+p}(s-a)^{\frac{1}{2}-p}}{s-x}(-\pi) = -\pi e^{-p\pi i}\frac{(b-s)^{\frac{1}{2}+p}(s-a)^{\frac{1}{2}-p}}{s-x} . \tag{18}$$

Likewise

$$\lim_{\varepsilon\to 0}\int_{C_{x^\pm}} F_p(t)\,dt = \pm\pi e^{\pm p\pi i}\frac{(b-x)^{\frac{1}{2}+p}(x-a)^{\frac{1}{2}-p}}{s-x} . \tag{19}$$

Taking into account (17) – (19), after the passage to the limit in equality (7) we obtain:

$$\int_a^b \left(\frac{b-t}{t-a}\right)^p \frac{dt}{(t-s)(t-x)}dt = -\pi tgp\pi\frac{(b-s)^{\frac{1}{2}+p}(s-a)^{\frac{1}{2}-p}}{s-x} + \pi tgp\pi\frac{(b-x)^{\frac{1}{2}+p}(x-a)^{\frac{1}{2}-p}}{s-x} - \frac{\pi}{\cos p\pi} . \tag{20}$$

Similarly

$$\int_a^b \left(\frac{b-t}{t-a}\right)^p \frac{\sqrt{(b-t)(t-a)}}{(t-s)(t-x)} dt = -\pi ctgp\pi \left(\frac{b-s}{s-a}\right)^p \frac{1}{s-x} - \pi ctgp\pi \left(\frac{b-x}{x-a}\right)^p \frac{1}{s-x}. \quad (21)$$

Substituting (20) and (21) into (2) b we arrive at the following equation:

$$\left(1-\lambda^2\right)\left(\frac{b-x}{x-a}\right)^p \varphi(x) + \left[\frac{ctgp\pi}{\pi}\int_a^b \left(\frac{b-t}{t-a}\right)^p \frac{\varphi(t)}{t-x} dt\right] -$$

$$-\frac{ctgp\pi}{\pi}\left(\frac{b-x}{x-a}\right)^p \int_a^b \frac{\varphi(t)}{t-x} dt + \left[\lambda^2 \frac{tgp\pi}{\pi}\int_a^b \left(\frac{b-t}{t-a}\right)^p \frac{\varphi(t)}{t-x} dt\right] -$$

$$-\lambda^2 \frac{tgp\pi}{\pi}\left(\frac{b-x}{x-a}\right)^p \int_a^b \frac{\sqrt{(b-x)(x-a)}}{(b-t)(t-a)} \frac{\varphi(t)dt}{t-x} + \frac{\lambda^2}{\pi\cos p\pi}\int_a^b \frac{\varphi(t)dt}{\sqrt{(b-t)(t-a)}} = (Af)(x). \quad (22)$$

Let us choose the parameter " p " such that

$$ctgp\pi = -\lambda^2 tgp\pi. \quad (23)$$

As a result, the terms contained in square brackets in equality (22) cancel each other out. Let us find an explicit match for the " p " parameter. From condition (23) we obtain

$$ctgp\pi = \pm\lambda i, \quad tgp\pi = \pm\frac{1}{\lambda i}, \quad \cos p\pi = \pm\frac{\lambda}{\sqrt{\lambda^2-1}}, \quad \Rightarrow \quad p_\pm = \frac{1}{2\pi i} Ln\frac{\pm\lambda+1}{\pm\lambda-1}. \quad (24)$$

In what follows, we will assume that $p = p_\pm$. Taking into account equalities (15) and after cancellation by $(b-x)^p (x-a)^{-p}$, equation (22) takes the form

$$\left(1-\lambda^2\right)\varphi(x) + \frac{\lambda}{\pi i}\int_a^b \frac{\varphi(t)}{t-x} dt -$$

$$-\frac{\lambda}{\pi i}\int_a^b \frac{\sqrt{(b-x)(x-a)}}{(b-t)(t-a)} \frac{\varphi(t)dt}{t-x} = \left(\frac{x-a}{b-x}\right)^p \left[(Af)(x) + C\right], \quad (25)$$

where

$$C = -\frac{\lambda\sqrt{\left(\lambda^2-1\right)}}{\pi}\int_a^b \frac{\varphi(t)dt}{\sqrt{(b-t)(t-a)}}.$$

Adding the sought equation to equation (25),

$$\frac{1}{\pi i}\int_a^b \frac{\varphi(t)}{t-x} dt - \lambda^2 \frac{1}{\pi i}\int_a^b \sqrt{\frac{(b-x)(x-a)}{(b-t)(t-a)}} \frac{\varphi(t)dt}{t-x} = f(x), \quad (26)$$

we obtain a system from which we exclude the integrals containing the expression $\sqrt{(b-x)(x-a)\left[(b-t)(t-a)\right]^{-1}}$. As a result, we get the equation

$$-\lambda\varphi(x) + \frac{1}{\pi i}\int_a^b \frac{\varphi(t)}{t-x} dt = \frac{1}{1-\lambda^2} f(x) - \frac{\lambda}{1-\lambda^2}\left(\frac{x-a}{b-x}\right)^p \left[(Af)(x) + C\right]. \quad (27)$$

Recall (see [4, 5]) that the solution to the equation

$$\mu\varphi(x) + \frac{1}{\pi i}\int_a^b \frac{\varphi(t)}{t-x} dt = g(x) \quad (28)$$

in the class of functions unbounded at $x = b$ and unbounded at $x = a$ has the form

$$\varphi(x) = \frac{\mu}{\mu^2-1} g(x) - \frac{1}{\pi i\left(\mu^2-1\right)}\left(\frac{b-x}{x-a}\right)^q \int_a^b \left(\frac{t-a}{b-t}\right)^q \frac{g(t)}{t-x} dt, \quad (29)$$

where

$$q = \frac{1}{2\pi i} Ln\frac{\mu-1}{\mu+1}, \quad \mu \notin (-\infty, -1] \cup [1, +\infty).$$

In the case of equation (18), we have $\mu = -\lambda \Rightarrow q = \dfrac{1}{2\pi i} Ln\dfrac{\lambda+1}{\lambda-1} = p$,

$$g(x)=\frac{1}{1-\lambda^2}f(x)-\frac{\lambda}{1-\lambda^2}\left(\frac{x-a}{b-x}\right)^p\left[\frac{1}{\pi i}\int_a^b\left(\frac{b-t}{t-a}\right)^p\frac{f(x)}{t-x}dt+C\right].$$

Substituting these expressions for q and $g(x)$ in (29), we obtain

$$\varphi(x)=\frac{\lambda}{1-\lambda^2}f(x)-\frac{\lambda^2}{\pi i\left(1-\lambda^2\right)^2}\left(\frac{x-a}{b-x}\right)^p\int_a^b\left(\frac{b-t}{t-a}\right)^p\frac{f(t)}{t-x}dt+\frac{1}{\pi i\left(1-\lambda^2\right)^2}\left(\frac{b-x}{x-a}\right)^p\int_a^b\left(\frac{t-a}{b-t}\right)^p\frac{f(x)}{t-x}dt-$$

$$-\frac{\lambda}{(\pi i)^2\left(1-\lambda^2\right)^2}\left(\frac{b-x}{x-a}\right)^p\int_a^b\left(\frac{t-a}{b-t}\right)^{2p}\frac{dt}{t-x}\int_a^b\left(\frac{b-s}{s-a}\right)^p\frac{f(s)}{s-t}ds-$$

$$-\frac{\lambda^2C}{\left(1-\lambda^2\right)^2}\left(\frac{x-a}{b-x}\right)^p-\frac{\lambda C}{\pi i\left(1-\lambda^2\right)^2}\left(\frac{b-x}{x-a}\right)^p\int_a^b\left(\frac{t-a}{b-t}\right)^p\frac{dt}{t-x}.$$

In the repeated singular integrals, we change the order of integration, using the Poincaré-Bertrand formulas, and we obtain

$$\varphi(x)=-\frac{\lambda^2}{\pi i\left(1-\lambda^2\right)^2}\left(\frac{x-a}{b-x}\right)^p\int_a^b\left(\frac{b-t}{t-a}\right)^p\frac{f(x)}{t-x}dt+\frac{1}{\pi i\left(1-\lambda^2\right)^2}\left(\frac{b-x}{x-a}\right)^p\int_a^b\left(\frac{t-a}{b-t}\right)^p\frac{f(t)}{t-x}dt-$$

$$-\frac{\lambda}{\pi^2\left(1-\lambda^2\right)^2}\left(\frac{b-x}{x-a}\right)^p\int_a^b\left(\frac{b-s}{s-a}\right)^p f(s)ds\left[\int_a^b\left(\frac{t-a}{b-t}\right)^{2p}\frac{dt}{(t-s)(t-x)}\right]-$$

$$-\frac{\lambda^2C}{\left(1-\lambda^2\right)^2}\left(\frac{x-a}{b-x}\right)^p-\frac{\lambda C}{\pi i\left(1-\lambda^2\right)}\left(\frac{b-x}{x-a}\right)^p\left[\int_a^b\left(\frac{t-a}{b-t}\right)^{2p}\frac{dt}{t-x}\right]. \tag{30}$$

We calculate the integrals from the square brackets in equality (21), obtaining

$$\int_a^b\left(\frac{t-a}{b-t}\right)^{2p}\frac{dt}{(t-s)(t-x)}=\pi\frac{\lambda^2+1}{2\lambda i}\left(\frac{s-a}{b-s}\right)^{2p}\frac{1}{s-x}-\pi\frac{\lambda^2+1}{2\lambda i}\left(\frac{x-a}{b-x}\right)^{2p}\frac{1}{s-x}. \tag{31}$$

$$\int_a^b\left(\frac{t-a}{b-t}\right)^p\frac{dt}{t-x}=-\pi i\frac{\lambda^2+1}{2\lambda}\left(\frac{x-a}{b-x}\right)^{2p}+\pi i\frac{\lambda^2-1}{2\lambda}. \tag{32}$$

Substituting these integrals into equality (30), we finally obtain a solution to equation (2) (for $m=1$).

$$\varphi(x)=\frac{1}{2\pi i\left(1-\lambda^2\right)}\left[\left(\frac{x-a}{b-x}\right)^p\int_a^b\left(\frac{b-t}{t-a}\right)^p\frac{f(x)}{t-x}dt+\left(\frac{b-x}{x-a}\right)^p\int_a^b\left(\frac{t-a}{b-t}\right)^p\frac{f(t)}{t-x}dt\right]+$$

$$+C_1\left[\left(\frac{x-a}{b-x}\right)^p+\left(\frac{b-x}{x-a}\right)^p\right]. \tag{33}$$

By substituting $\varphi(x)$ into the desired equation, we make sure that the terms in the first row form a particular solution of the inhomogeneous equation, and the terms of the second row form the solution to the homogeneous equation, where C_1 is an arbitrary constant.

For the convergence of integrals (31) and (32), it is necessary that $\operatorname{Re}2p=2\operatorname{Re}p<1$, i.e. $\operatorname{Re}p<\frac{1}{2}$. Since $\operatorname{Re}p=\frac{1}{2\pi}\arg\frac{\lambda+1}{\lambda-1}$, then the condition $\operatorname{Re}p<\frac{1}{2}$ is satisfied for $0<\arg\frac{\lambda+1}{\lambda-1}<\pi$. The last condition is satisfied for all λ for which $\operatorname{Re}\lambda<0$. However if we put $p=p_-$ in (24), then we arrive at the condition $0<\arg\frac{\lambda-1}{\lambda+1}<\pi$. This condition is satisfied for all λ for which $\operatorname{Re}\lambda<0$, i.e, the imaginary axis must be excluded from the $\lambda-$plane.

In the case of kernels K_2, K_3 and K_4, by the same method, Eq. (2) is reduced to Eq. (27) with the same difference as $C=0$, and to the same solution (33) with $C_1=0$.

However, in the case of kernel K_4, in general, formula (33) is not a solution unless the condition

$$\int_a^b \left[\left(\frac{x-a}{b-x} \right)^p + \left(\frac{b-x}{x-a} \right)^p \right] f(t)\,dt = 0$$

is satisfied. This condition means that $f(t)$ must be orthogonal to the solution of the homogeneous equation. In the case of K_2 and K_3 the solution is unique ($C_1 = 0$) and unconditional.

Due to the behavior of the Cauchy-type integral at the ends of the contour of integration [5, 6], solution (33) belongs to the class of functions that are unbounded at $x = a$ and $x = b$.

Conclusions. The methods used in this work, integral transformations, analytical continuation, after a successful choice of the integration contour, the use of Cauchy theory of residues and limit transition allowed the solution of this integral equation to be reduced to the solution of some characteristic singular equation. These methods can be used in solving other singular equations, as well as in calculating some singular integrals that depend on parameters.

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Надійшла (received) 24.03.2025

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