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EFFECTIVE DIVERSIFICATION AND STOCK PORTFOLIO DYNAMICS UNDER CONSTRAINTS

This scientific study considers the mathematical problem of optimal diversification of a portfolio of shares in the presence of market constraints. Mathematical formulation of the constructing trajectory problem of one share the market value is given in the class of ordinary first-order differential equations. The procedure for constructing a general solution of such an equation is given. Of particular practical importance is the mathematical problem of constructing an optimal portfolio structure in the presence of quantitative and qualitative market constraints. Such constraints arise at every moment of portfolio diversification and their consideration significantly complicates the problem. The procedure for building a dynamic model of the formation of the market value of one share is based on the application of the market model of W. Sharpe and the fundamental theory by H. Markowitz. The principles of H. Markowitz theory make it possible to determine the optimal values of the portfolio's expected profitability and riskiness when applying the procedure for building an optimal portfolio of risky securities. The application of optimal management theory methods in the optimization of the stock portfolio involves an iterative procedure for determining the optimal structure. The work also considers an important applied problem of applying the theory of H. Markowitz to solve the problem of optimal diversification of a portfolio of risky investments in the presence of restrictions that are formed by the stock market at each moment of time. The presence of market restrictions significantly affects the decision-making procedure regarding optimal portfolio diversification. This scientific study presents an algorithm for optimal diversification of a portfolio of risky securities in the presence of market restrictions.

Key words: portfolio optimization, fundamental analysis, portfolio diversification, effective set.

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ЄФЕКТИВНА ДИВЕРСИФІКАЦІЯ ТА ДИНАМІКА ПОРТФЕЛЯ АКЦІЙ ЗА УМОВ ОБМЕЖЕНЬ

У даному науковому дослідженні розглядається математична задача оптимальної диверсифікації портфеля акцій за наявності ринкових обмежень. Така прикладна задача є актуальною проблемою інвестування у ризиковані активи і до її розв’язання прикута увага провідних математиків та економістів в усьому світі. Серед найбільш відомих можна назвати Г. Марковиця, Д. Тобіна, К. Іто, Ф. Блек, М. Шоулз, Р. Мертон. Математичну постановку задачі побудови траєкторії руху однієї акції за ринковою вартістю наведено в класі звичайних диференціальних рівнянь і побудовано, спираючись на ринкову модель В. Шарпа. Особливе практичне значення має математична задача побудови оптимальної структури портфеля за наявності кількісних та якісних ринкових обмежень. Такі обмеження виникають при диверсифікації портфеля і необхідність їх врахування суттєво ускладнює задачу. Процедура побудови динамічної моделі формування ринкової вартості однієї акції базується на застосуванні ринкової моделі В. Шарпа та фундаментальної теорії Г. Марковіца. У роботі також розглядається важлива прикладна проблема застосування теорії Г. Марковіца для розв’язання задачі оптимальної диверсифікації портфеля ризикованих інвестицій за наявності обмежень, що формуються фондовим ринком. Такий підхід враховує побудову та застосування допустимої та ефективної множин портфелів цінних паперів. Ці множини, згідно Г. Марковицю, формуються на основі наявних на ринку активів. Наявність ринкових обмежень суттєво впливає на процедуру прийняття рішень щодо оптимальної диверсифікації портфеля. У цьому науковому дослідженні наведено алгоритм оптимальної диверсифікації портфеля ризикованих цінних паперів за наявності обмежень.

Ключові слова: оптимізація, фундаментальний аналіз, диверсифікація портфеля, ефективна множина.

Introduction. The problem of mathematical description of the dynamics of market value formation of one share and a portfolio of shares has long been relevant and attracts the attention of scientists and practitioners. According to *H. Markowitz*, mathematically, such a problem is formulated as a two-criterion problem of nonlinear optimization. The criteria are mutually contradictory and it does not have a single solution. Despite this, its practical importance is extremely great. Taking into account the complexity of mathematical statements and their practical implementation, researchers have identified important principled approaches to decision-making on optimizing the structure of the investment portfolio. Among such approaches, the most famous can be distinguished:

– H. Markowitz approach, using a valid and effective set, investor indifference curves. The theory of H. Markowitz is classic and its principles are the basis of many decision-making strategies in the stock market.

– Splitting two-criterion investment portfolio optimization problem into two one-criterion. The mathematical two-criterion problem of optimizing a portfolio of risky securities involves maximizing the expected return and minimizing the risk of the investment portfolio.

– Methods of technical analysis. The most common technology for making practical decisions involves the use of known statistical information about the dynamics of the market value of one share.

– Methods of fundamental analysis. The theory of fundamental analysis is actively developing and includes new mathematical models and methods for describing the dynamics and procedures for making decisions about the optimal structure of an investment portfolio. In the researches of *K. Ito*, *F. Black*, *M. Scholes*, and *R. Merton*, new fundamental results are formulated. The results are actively developed and implemented in the practice of decision-making on the stock market.

In this study an algorithm for identifying of mathematical model parameters of market value dynamics of one share and a portfolio of shares is proposed. The principle of operation of the algorithm is based on the application of an iterative procedure, at each step of which model parameters are calculated that improve the value of the selected quality criterion. It is worth noting that the choice of model parameter optimization procedure may depend on the selected quality criterion.

Development of Portfolio Investment Theory. The portfolio model was originally proposed in [1, 2]. Based on the relationship between income and risk, H. Markowitz constructs the mean-variance model. He put forward the risk measurement and the modern principles of portfolio investment. This has laid a theoretical foundation for the researches of securities investment portfolio. Sharpe's capital asset pricing model have been known and are used by many investors when solving the problems of investing in shares [3, 4].

Among other important approaches to solving applied portfolio theory problems, the following can be noted:

– qualitative and quantitative analysis of investment management models [5, 6];
– evolutionary and memetic computing for project portfolio selection [7, 8];
– an exploration of meta-heuristic approaches for the project portfolio selection [9];
– a simheuristic methods for project portfolio selection under uncertainty [10, 11];

Construction of a mathematical model of a stock portfolio. Let's consider a mathematical model of the market-value of the investment portfolio and build appropriate analytical trajectory. At the time interval $t \in [t_0, T]$ equation describing the profitability of the portfolio shares r_p, has the form

$$r_p(t) = \sum_i x_i(t) r_i(t), \quad (1)$$

where x_i – proportion of shares and type of portfolio; r_i – expected return on equity. Having differentiated both parts of (1), we get

$$\frac{dr_p(t)}{dt} = \sum_i \left(r_i(t)\frac{dx_i(t)}{dt} + x_i(t)\frac{dr_i(t)}{dt} \right). \quad (2)$$

Given that there is a property for $i \neq j$ and $\sum_j b_j = 1$, $\sum_i a_i b_i = \sum_i a_i + \sum_i \sum_j a_i b_j$.

In this way occurring ratio

$$\sum_i x_i(t) r_i(t) \frac{f_i}{r_i(t)} = \sum_i x_i(t) r_i(t) - \sum_i \sum_j x_i(t) \times r_i(t) \frac{f_j}{x_j(t)},$$

$$\sum_i \frac{x_i(t) r_i(t)}{x_i(t)} \frac{dx_i(t)}{dt} = \sum_i x_i(t) r_i(t) - \sum_i \sum_j x_i(t) \times r_i(t) \frac{dx_j(t)}{dt} \frac{1}{x_j(t)}.$$

The dynamic model of the market value of the stock portfolio will have the following form

$$\frac{dr_p(t)}{dt} = 2r_p(t) - \sum_i \sum_j x_i(t) r_i(t) \left(\frac{f_j}{r_j(t)} \times \frac{dx_j(t)}{dt} \frac{1}{x_j(t)} \right). \quad (3)$$

This is a first order linear differential equation. Applying the method of variation of arbitrary constants and getting its general solution

$$r_p(t) = -\ell^{2t} \int \ell^{-2t} \sum_i \sum_j x_i(t) r_i(t) \left(\frac{f_j}{r_j(t)} \times \frac{dx_j(t)}{dt} \frac{1}{x_j(t)} \right) dt, \quad (4)$$

where

$$f_j(t) = \left(\alpha_1 SM_{ind}(t) + \alpha_2 I(t) \right) r_j(t) + \sum_{i=1}^{N} \beta_{ij} r_j(t).$$

In latter ratio the assumptions made phenomenon that describes the dynamics of the formation of the market value of the portfolio of risky securities. A more detailed analysis indicates two important properties that characterize the market value of the portfolio: dynamics depends on the dynamics of both the expected profitability of shares and changes in the structure of the portfolio. In relation (3) f_j is the right-hand side of the differential equation that describes the dynamics of the formation of the market value of the shares [5]. Its structure and content corresponds to a market model of *W. Sharpe* [3].

We were able to construct a sequence paths at selected time intervals, allowing you to create the initial structure of the investment portfolio. However, this approach can not effectively influence the structure of investment and not fully using the mathematical description of the properties listed in mathematical models (4).

Among other, it contains a factor

$$\frac{dx_j(t)}{dt} * \frac{1}{x_j(t)},$$

that describes the possible changes in the structure of investment. To simplify further calculations mathematical model (4) is presented in more general terms

$$\dot{r}_p = f^p\left(r_p, x_i, \dot{x}_i, r_i, \dot{r}_i\right), \ i = \overline{1, l}. \quad (5)$$

Optimal portfolio diversification. In this study, attention is focused on the possibility of applying the theory of effective and admissible sets by H. Markowitz for the diversification of the portfolio of risky investments. The real market involves a dynamic change in the market values of shares, which prompts the investor to actively diversify the portfolio. According to the algorithm for constructing the admissible set, the portfolio that is more effective in terms of expected profitability and riskiness for the investor will be located higher and to the left on the "expected profitability – riskiness" plane. According to H. Markowitz, portfolios that are elements of the effective set will be optimal. Such portfolios are Pareto-optimal. The classic problem of portfolio optimization is formulated without taking into account restrictions on changing the portfolio structure. Such a feature significantly affects the possibilities of real investment. The algorithms proposed in this study make it possible to take into account such limitations and form a portfolio that is closest to the optimal one that is potentially possible at the moment on the market. The developed method can be effectively applied to computer programs that perform automated selection of the optimal portfolio structure.

We consider the problem of optimal portfolio diversification under constraints. Let's move on to the second problem in the general formulation of H. Markowitz about optimizing the risk of the portfolio of shares that is optimal in terms of expected profitability. For this, we will use sets of admissible and efficient portfolios corresponding to the selected set of shares. The risk optimization procedure for the optimal expected return portfolio consists in choosing at each step admissible portfolios that lie on the EF line. This line connects point E, which corresponds to the optimal market value of the portfolio, and point F, which belongs to the efficient set. This line is parallel to the portfolio's riskiness axis. The peculiarity of this selection of the optimal portfolio is that on this straight line, according to the definition, each of the portfolios corresponds to the same expected return, but the riskiness decreases in the direction of the axis. This property of the admissible set of investment portfolios allows, on the one hand take into account the restrictions

$$x_i(t) \in X(t),\ i = \overline{1, n}$$

and on the other hand – to determine the portfolio of "optimal" expected return with less risk.

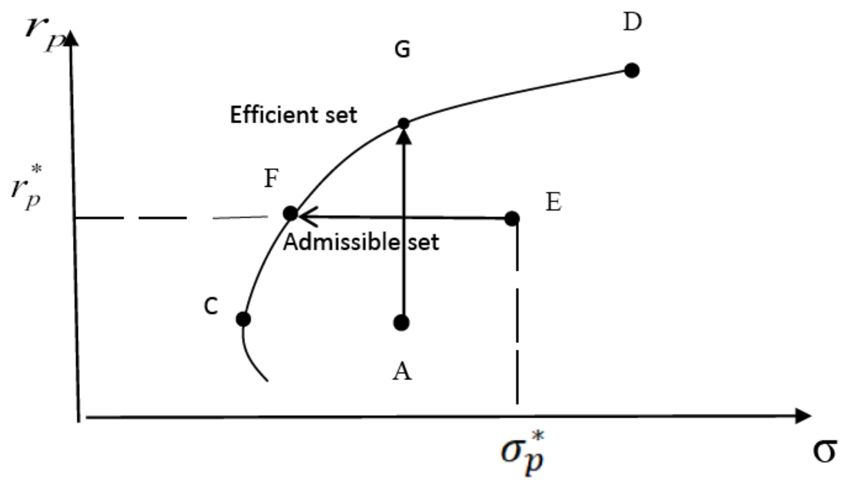

Fig. 1 – Admissible and efficient sets of portfolios of risky securities.

If the specified portfolio is located at point E, that is, it is one for which it is not possible to reduce the riskiness according to the rule proposed above, then the "optimal portfolio" is determined by moving it from E, to F, which is an element of the effective set of portfolios. In effect, this means identifying a portfolio of stocks with higher expected returns. At the same time, this procedure allows you to constructively take into account existing limitations when diversifying the portfolio. Another mathematical formulation of the problem of optimization of the expected profitability $r_p(T)$ of the investment portfolio at a certain time T level of its risk τ is as follows

$$\begin{cases} r^{\mathrm{T}}(T)x(T) \to \max\limits_x \\ x^{\mathrm{T}}(T)Vx(T) = \tau \\ I^{\mathrm{T}}x(T) = 1 \\ x_i(t) \ge 0, i = \overline{1,n}, t \in [t_0, T] \\ x_i(t) \in X(t), i = \overline{1,n}, t \in [t_0, T] \end{cases}.$$

The procedure for optimizing the portfolio's expected return r_p for a certain level of its risk consists in choosing at each step admissible portfolios that lie on the line AG connecting point A, which corresponds to the optimal portfolio calculated by expected return, and point G, which belongs to the efficient set. This line is parallel to the axis of market value r_p. The peculiarity of this selection of the optimal portfolio is that on this straight line, according to the definition, each of the portfolios corresponds to the same riskiness, but the market value r_p increases. This property of the admissible set of investment portfolios, as in the previous case, allows, on the one hand, to take into account restrictions $x_i(t) \in X(t)$, $i = \overline{1, n}$ and on the other hand, to determine the portfolio with the "optimal" risk and higher expected profitability.

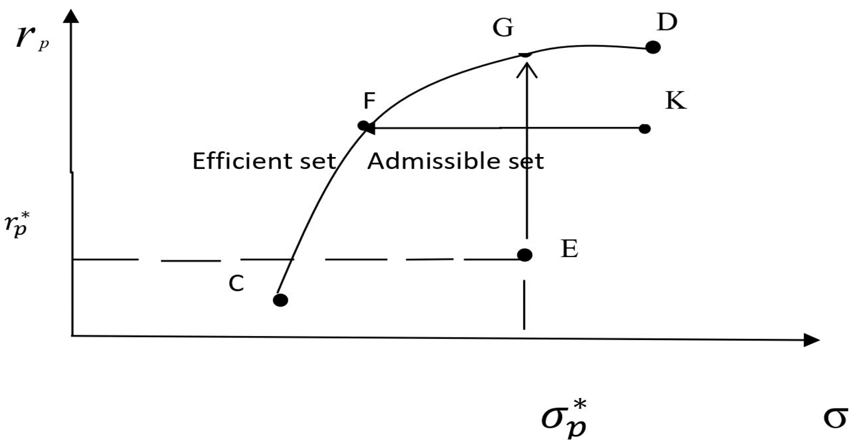

Fig. 2 – Optimizing the market value of the stock portfolio.

If the defined portfolio is located at point K , that is, one for which there is no possibility to increase the expected return, according to the rule proposed above, then the "optimal portfolio" is determined by moving it from K to F, which is an element of the effective set of portfolios. In fact, this means reducing the riskiness of the stock portfolio. The effective set or the set of effective portfolios in figures 1, 2 is on the arc. It is a Pareto set [5] for a set of shares existing on the market.

Prospects for further research. The effectiveness of the application of the above methods may be associated with the construction of an adequate mathematical model of the dynamics of the market value of the portfolio. The paper presents one of the possible procedures for constructing such a model. Further research should focus on the development of approaches to clarifying the structure and parameters of the mathematical model (4).

Prospects for research in this applied field can also be linked to the combination of technical and fundamental analysis methods together with artificial intelligence approaches. Such attempts have been made in the works [7, 8], as well as in the studies of other authors. This study is one of the attempts to effectively combine these approaches to decision-making when investing in securities.

Conclusion. In this study new mathematical formulations of the optimization problems of the stock portfolio structure are given and methods of their solution are developed. Mathematical problems formulated on the basis of models of the dynamics of the market value of one share and a portfolio of shares make it possible to solve the problem of optimal diversification of the investment portfolio, taking into account quantitative and qualitative market restrictions on the portfolio structure.

The formulation of the problem and built algorithm greatly expand the possibilities of investing, as in every moment of diversification it is possible to construct a set of alternatives that are equivalent in terms of selected quality criteria. The investor, as in the classical approach, Mr. Markowitz has the ability to take into account when deciding additional factors that arise in the course of practical investment and are associated with the peculiarities of the system dynamics modeling. Mathematical portfolio diversification procedure makes it possible for the above-mentioned models of the dynamics of the market value of a one share and stock portfolio to solve the problem of portfolio diversification.

Bibliography

1. *Markowitz H.* Portfolio selection // Journal of Finance. – 1952. – vol. 7. – no. 1. – pp. 77 – 91. DOI: 10.1111/j.1540-6261.1952.tb01525.x.
2. *Markowitz H.* Portfolio selection: Efficient diversification of investments. – John Wiley & Sons, New York, 1959.
3. *Sharpe W.* Capital asset prices: A theory of market equilibrium under conditions of risk // Journal of Finance. – 1964. – vol. 19. – issue 3. – pp. 425 – 442. DOI: 10.1111/j.1540-6261.1964.tb02865.x.
4. *Sharpe W.*, *Alexander G.*, *Bailey J.* Investments. – Prentice Hall, New Jersey, 1998. – 962 p.
5. *Kulian V.*, *Yunkova O.*, *Korobova M.* Mathematical problem of banking assets diversification // Bulletin of Taras Shevchenko National University of Kyiv. – 2021. – no.1. – pp. 85 – 88. DOI: 10.17721/1812-5409.2021/1.11.
6. *Kulian V.*, *Yunkova O.*, *Korobova M.* Solutions sensitivity when modeling of investment dynamics // Bulletin of Taras Shevchenko National University of Kyiv. – 2022. – no. 4. – pp. 51 – 54. DOI: 10.17721/1812-5409.2022/4.6.
7. *Kolisch R.*, *Fliedner T.* Evolutionary and Memetic Computing for Project Portfolio Selection and Scheduling. – Springer, Berlin, 2022. DOI: 10.1007/978-3-030-88315-7.
8. *Mohagheghi V.*, *Mousavi S.*, *Mojtahedi M.* Project portfolio selection problems: Two decades review from 1999 to 2019 // Journal of Intelligent

and Fuzzy Systems. – 2020. – vol. 38(4): 1 – 15. DOI: 10.3233/JIFS-182847.

9. *Harrison K. R., Elsayed S., Weir T., Garanovich I. L., Taylor R., Sarker R.* An exploration of meta-heuristic approaches for the project portfolio selection and scheduling problem in a defence context // in: Proceedings of the 2020 IEEE Symposium Series on Computational Intelligence (SSCI). – IEEE, Canberra, 2020. – pp. 1395 – 1402. DOI: 10.1109/ssci47803.2020.9308608.
10. *Panadero J., Doering J., Kizys R., Juan A. A., Fito A.* A variable neighborhood search simheuristic for project portfolio selection under uncertainty // Journal of Heuristics. – 2020. – vol. 26. – pp. 1 – 23. DOI: 10.1007/s10732-018-9367-z.
11. *Kar M. B., Kar S., Guo S., Li X., Majumder S.* A new bi-objective fuzzy portfolio selection model and its solution through evolutionary algorithms, // Soft Computing. – 2019. – vol. 23. – pp. 4367 – 4381. DOI: 10.1007/s00500-018-3094-0.

References (transliterated)

1. Markowitz H. Portfolio selection. *Journal of Finance*. 1952, vol. 7, no. 1, pp. 77–91. DOI: 10.1111/j.1540-6261.1952.tb01525.x.
2. Markowitz H. *Portfolio selection: Efficient diversification of investments*. John Wiley & Sons, New York, 1959.
3. Sharpe W. Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance*. 1964, vol. 19, issue 3, pp. 425–442. DOI: 10. 1111/j.1540-6261.1964.tb02865.x.
4. Sharpe W., Alexander G., Bailey J. Investments. – Prentice Hall, New Jersey, 1998. – 962 p.
5. Kulian V., Yunkova O., Korobova M. Mathematical problem of banking assets diversification. *Bulletin of Taras Shevchenko National University of Kyiv*. 2021, no. 1, pp. 85–88. DOI: 10.17721/1812-5409.2021/1.11.
6. Kulian V., Yunkova O., Korobova M. Solutions sensitivity when modeling of investment dynamics. *Bulletin of Taras Shevchenko National University of Kyiv*. 2022, no. 4, pp. 51–54. DOI: 10.17721/1812-5409.2022/4.6.
7. Kolisch R., Fliedner T. *Evolutionary and Memetic Computing for Project Portfolio Selection and Scheduling*. Springer, Berlin, 2022. DOI: 10. 1007/978-3-030-88315-7.
8. Mohagheghi V., Mousavi S., Mojtahedi M. Project portfolio selection problems. Two decades review from 1999 to 2019. *Journal of Intelligent and Fuzzy Systems*. 2020, vol. 38(4): 1 – 15. DOI: 10.3233/JIFS-182847.
9. *Harrison K. R., Elsayed S., Weir T., Garanovich I. L., Taylor R., Sarker R.* An exploration of meta-heuristic approaches for the project portfolio selection and scheduling problem in a defence context. *in: Proceedings of the 2020 IEEE Symposium Series on Computational Intelligence (SSCI)*. IEEE, Canberra, 2020. pp. 1395–1402. DOI: 10.1109/ssci47803.2020.9308608.
10. Panadero J., Doering J., Kizys R., Juan A. A., Fito A. A variable neighborhood search simheuristic for project portfolio selection under uncertainty. *Journal of Heuristics*. 2020, vol. 26, pp. 1–23. DOI: 10.1007/s10732-018-9367-z.
11. Kar M. B., Kar S., Guo S., Li X., Majumder S. A new bi-objective fuzzy portfolio selection model and its solution through evolutionary algorithms. *Soft Computing*. 2019, vol. 23, pp. 4367–4381. DOI: 10.1007/s00500-018-3094-0.

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